Lecture 14: Fourier Transforms

Definition-

1. $\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(x) e^{-ikx} dx}_{\text{Projection onto k-th mode}}$ (Fourier Transform)

Amplitude of k-th mode

2. $\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk$ (Inverse Fourier Transform)

Existence- If $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ then $\hat{f}(k)$ is well defined.

proof:

$$|\mathcal{F}[f](k)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx$$

$$< \infty.$$

$\blacksquare$

Examples:

1. $f(x) = e^{-ax^2}$, $(a > 0)$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x^2 + ik/a\, x)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x^2 + ik/a\, x - k^2/4a^2)} e^{-k^2/4a} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-a(x + ik/2a)^2} dx$$

$$= \frac{1}{\sqrt{2a}} e^{-k^2/4a}.$$

2. $f(x) = e^{-|x|}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{-ikx}\,dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{1+k^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$$

3. $f(x) = \begin{cases} 1, & |x| < a \\ 0, & \text{o.w} \end{cases} \qquad (a > 0)$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\,dx$$

$$= \frac{-1}{\sqrt{2\pi}} \cdot \frac{1}{ik} \left(e^{-ika} - e^{ika} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(ka)}{k}$$

Properties:

1. $\mathcal{F}[af](k) = a\,\mathcal{F}[f](k)$
2. $\mathcal{F}[f(ax)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-ikx}\,dx$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a} \int_{-\infty}^{\infty} f(x) e^{-ikx/a}\,dx$$

$$= \frac{1}{a} \mathcal{F}[f]\left(\frac{k}{a}\right)$$

3. $\mathcal{F}[f](k) = \mathcal{F}^{-1}[f](-k)$
4. $\mathcal{F}[f(x-a)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{-ikx}\,dx$

$$= \frac{1}{\sqrt{2\pi}} e^{-ika} \int_{-\infty}^{\infty} f(u) e^{-iku}\,du$$

$$= e^{-ika} \mathcal{F}[f](k).$$

Derivatives and Integrals

1. $\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx$

$$= \frac{(ik)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

$$= ik\, \mathcal{F}[f](k).$$

2. $\mathcal{F}[f''(x)] = -k^2 \mathcal{F}[f](k).$

3. Let $F(x) = \int_{-\infty}^{x} f(x) dx$ (antiderivative)

$$\Rightarrow \mathcal{F}[F(x)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ikx} dx$$

$$= -\frac{1}{\sqrt{2\pi}\, ik} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

$$= \frac{i}{k} \mathcal{F}[f](k).$$

Convolutions:

The convolution of $f(x)$ and $g(x)$ is defined by

$$h(x) = f * g = \int_{-\infty}^{\infty} f(x-y) g(y) dy.$$

Theorem:

If $h(x) = (f*g)(x)$ then

1. $\hat{h} = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}.$

Also if $h(x) = f(x) g(x)$ then

2. $\hat{h} = \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(k)$

~~proof~~:

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-ikx} dy\, dx$$

$$\Rightarrow \hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{-ikx}\,dx\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(y)e^{-ik(u+y)}\,du\,dy$$

$$= \sqrt{2\pi}\,\hat{f}(k)\cdot\hat{g}(k).$$

Green's Function:

Find the Green's function for:

$$-\frac{d^2u}{dx^2} + w^2 u = f(x)$$

$$\lim_{|x|\to\infty} u(x) = 0$$

The Green's function satisfies

$$-G''(x,y) + w^2 G(x,y) = \delta(x-y)$$

Take Fourier transform:

$$+k^2\hat{G} + w^2\hat{G} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\delta(x-y)\,dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-iky}$$

$$\Rightarrow \hat{G} = \frac{1}{\sqrt{2\pi}}\frac{e^{-iky}}{w^2+k^2}.$$

$$\Rightarrow G = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left[\frac{e^{-iky}}{w^2+k^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left[\frac{e^{-iky}}{\sqrt{2\pi}}\right] * \mathcal{F}^{-1}\left[\frac{1}{w^2+k^2}\right]$$

$$= \frac{1}{2w}\delta(x-y) * \sqrt{\frac{\pi}{2}}\cdot\frac{1}{w}e^{-w|x|}$$

$$= \frac{1}{2w}\int_{-\infty}^{\infty}\delta(x-y-\xi)\cdot e^{-w|x|}\,d\xi$$

$$= \frac{1}{2w}e^{-w|x-y|}.$$

Therefore,

$$u(x) = \frac{1}{2w}\int_{-\infty}^{\infty} e^{-w|x-y|} f(y)\,dy$$

$$= \frac{1}{2w} e^{-w|x|} * f(x)$$

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Example:

Solve:

$$u_t = u_{xx}$$
$$u(0,x) = f(x)$$

Take Fourier transforms:

$$\hat{u}_t = -k^2 \hat{u}$$
$$u(0,k) = \hat{f}(k)$$

$$\Rightarrow \hat{u}(t,k) = \hat{f}(k)\, e^{-k^2 t}$$
$$\Rightarrow u(t,x) = \mathcal{F}^{-1}\left[\hat{f}(k)\, e^{-k^2 t}\right]$$

$$= \frac{1}{\sqrt{2\pi}} f(x) * \mathcal{F}^{-1}\left[e^{-k^2 t}\right]$$
$$= \frac{1}{\sqrt{2t}} f(x) * e^{-x^2/4t}$$
$$= \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{\sqrt{2t}} f(y)\,dy$$

The Green's function for the heat equation is

$$G(x,y) = \frac{1}{\sqrt{2t}} \exp\left(\frac{-(x-y)^2}{4t}\right).$$

This function is also called the heat kernel.