# Lecture 15: PDEs On Unbounded Domains

**Example:**

Solve $\quad u_t = u_{xx}$

$\quad\quad\quad u(0,x) = f(x)$

$$\Rightarrow \hat{u}_t = -k^2 \hat{u}$$
$$\hat{u}(0,k) = \hat{f}(k)$$

$$\Rightarrow \hat{u}(t,k) = \hat{f}(k) e^{-k^2 t}$$

$$\Rightarrow u(t,x) = \mathcal{F}^{-1}\left[\hat{f}(k) e^{-k^2 t}\right]$$

$$= \frac{1}{\sqrt{2\pi}} f(x) * \frac{1}{\sqrt{2t}} e^{-x^2/4t}$$

$$\Rightarrow \boxed{u(t,x) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}\, dy.}$$

$$G(x,y;t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right)$$

- Fundamental Solution
- Green's function
- Heat Kernel

**Properties:**

1. $$\int_{-\infty}^{\infty} G(x,y;t)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} dv$$

$$= 1.$$

2. If $x \neq y$, $\lim_{t \to 0} G(x,y;t) = 0.$

3. If $x = y$, $\lim_{t \to 0} G(x,y;t) = \infty$

Therefore,

$$\lim_{t \to 0} G(x,y;t) = \delta(x-y)$$

**Uniqueness:**

Solutions to $U_t = U_{xx}$, $U(0,x) = f(x)$ are unique.

*proof:* Let $E(t) = \int_{-\infty}^{\infty} U(t,x)^2 dx$.

$$\Rightarrow \frac{dE}{dt} = \int_{-\infty}^{\infty} 2U \cdot U_t \, dx$$
$$= \int_{-\infty}^{\infty} 2U \cdot U_{xx} \, dx$$
$$= -2\int_{-\infty}^{\infty} U_x^2 \, dx \le 0.$$

If $U_1, U_2$ solve the initial value problem then, $U_1 - U_2$ is also a solution. Therefore, $E[U_1 - U_2](t) = 0$ $\Rightarrow U_1 = U_2$.

**Boundedness of Solutions:**

Suppose $u$ solves

$$U_t = U_{xx}$$
$$u(0,x) = f(x),$$

then,

$$|u(t,x)| \le \max_{-\infty < x < \infty} |f(x)| = M$$

*proof:*

$$|u(t,x)| = \left| \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x) \exp\left(\frac{-(x-y)^2}{4\pi t}\right) dy \right|$$

$$\le \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(x)| \exp\left(\frac{-(x-y)^2}{4\pi t}\right) dy$$

$$\le M \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4\pi t}\right) dy$$

$$= M.$$

## Decay of Solutions:

If $u(t,x)$ solves

$$u_t = u_{xx}$$
$$u(0,x) = f(x)$$

and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ then

$$\lim_{t\to\infty} u(t,x) = 0.$$

~~proof~~:

$$|u(t,x)| = \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left( \frac{-(x-y)^2}{4t} \right) f(y)\,dy \right|$$

$$\leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(y)|\,dy.$$

$$\Rightarrow \lim_{t\to\infty} |u(t,x)| = 0.$$

$\Rightarrow$ Solutions to heat equation dissipate to nothing.

## Duhamel's Principle:

### Example 1:

How do we solve

$$y'(t) + ay = F(t), \quad y(0) = 0 \qquad (1)$$

Multiply by integrating factor $e^{at}$,

$$\frac{d}{dt}(e^{at}y) = e^{at}F(t)$$

$$\Rightarrow e^{at}y = \int_0^t e^{a\tau}F(\tau)\,d\tau \qquad (2)$$

$$\Rightarrow y(t) = \int_0^t e^{-a(t-\tau)}F(\tau)\,d\tau.$$

## Example 2:

Solve

$$w'(t;\tau)+aw(t;\tau)=0$$
$$w(0;\tau)=F(\tau)$$

$$\Rightarrow \frac{d}{dt}\left(e^{at}w(t;\tau)\right)=0$$

$$\Rightarrow w(t;\tau)=e^{-at}F(\tau)$$

*From example 1 and 2:

$$y(t)=\int_0^t e^{-a(t-\tau)}F(\tau)d\tau$$

$$=\int_0^t w(t-\tau;\tau)d\tau$$

Duhamel's Principle – The solution of the problem

$$y'+ay=F(t),\ t>0,\ y(0)=0$$

is given by

$$y(t)=\int_0^t w(t-\tau;\tau)d\tau$$

where $w=w(t,\tau)$ solves the homogeneous equation

$$w'(t;\tau)+aw(t;\tau)=0$$
$$w(0;\tau)=F(\tau)$$

## Example:

Lets try using Duhamel's principle to solve

(1) $U_t=U_{xx}+f(t,x)$

$u(0,x)=0$

Consider the homogeneous equation:

(2) $W_t=U_{xx}$

$w(x,0,\tau)=f(x,\tau)$

$$\Rightarrow w(x,t,\tau)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left(\frac{-(x-y)^2}{4t}\right)f(y,\tau)dy$$

From Duhamel's principle the solution to (1) should be:

$$\boxed{u(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-\tau)}}\exp\left(\frac{-(x-y)^2}{4(t-\tau)}\right)f(y,\tau)\,dy\,d\tau}$$

Why does this work?

$$u_t = u_{xx}+f(x,t)$$

$$\Rightarrow \hat{u} = -k^2\hat{u}+\hat{f}(k,\tau)$$

$$\hat{u}(k,0)=0$$

We solve the O.D.E. using Duhamel's principle.

$$w(k,t;\tau)_t = -k^2 w(k,t;\tau)$$

$$w(k,0;\tau)=\hat{f}(k,\tau)$$

$$\Rightarrow w(k,t;\tau)=\hat{f}(k,\tau)e^{-k^2t}$$

$$\Rightarrow \hat{u} = \int_0^t w(k,t-\tau;\tau)\,d\tau$$

$$=\int_0^t \hat{f}(k,\tau)e^{-k^2(t-\tau)}\,d\tau$$

$$\Rightarrow u(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{f(y,\tau)}{\sqrt{4\pi(t-\tau)}}\exp\left(\frac{-(x-y)^2}{4(t-\tau)}\right)dy\,d\tau$$

Example:

$$u_t + cu_x = f(x,t)$$

$$u(x,0)=0$$

Auxiliary problem

$$w_t + cw_x = 0$$

$$w(x,0;\tau)=f(x,\tau)$$

$$\Rightarrow w(x,t;\tau)=f(x-ct,\tau)$$

Duhamel's principle:

$$u(x,t)=\int_0^t w(x,t-\tau;\tau)\,d\tau$$

$$\Rightarrow u(x,t)=\int_0^t f(x-ct+c\tau,\tau)\,d\tau$$

Example:

$$u_t + 3u_x = x\cos^2(t)$$
$$u(x,0) = e^{-x^2}$$

By linear superposition:

$$u(x,t) = e^{-(x-3t)^2} + \int_0^t (x-3t+3\tau)\cos^2(\tau)\,d\tau$$

$$= \exp(-(x-3t)^2) - \frac{3}{8} - \frac{3}{4}t^2 + \frac{tx}{2} + \frac{3}{8}\cos(2t) + \frac{x\sin(2t)}{4}$$