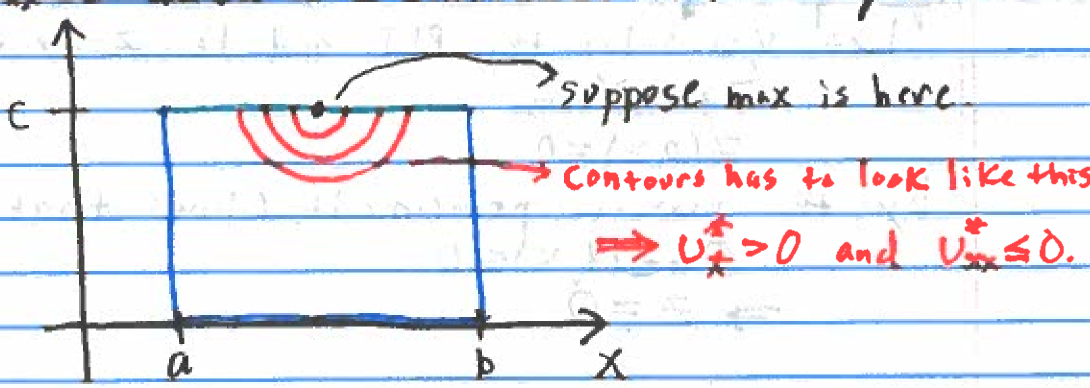
Lecture 16: The Maximum Principle

Theorem - Let $\gamma > 0$. Suppose $u(t,x)$ is a solution to
$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + F(t,x)$$
on the domain
$$R = \{a < x < b,\ 0 < t < c\}.$$
Assume that $F(t,x) \le 0$, the the maximum of $u(t,x)$ is obtained at $t=0$ or $x=a$ or $x=b$.

proof:

1. Assume $F < 0$. Then,
$$u_t < \gamma u_{xx}.$$
Let (t^*, u^*) denote a local maximum. If (t^*, u^*) lies in the interior of R it follows that $u_t^* = u_x^* = 0$. By the second derivative test $u_{xx}^* \le 0$. This is not possible so the maximum must occur on the boundary of R.

To exclude the case $t = c$, we note that $u_t^* \ge 0$ and $u_{xx}^* \le 0$ which is a contradiction (see figure).



2. We now need to consider the case $F \le 0$. Let
$$v(t,x) = u(t,x) + \varepsilon x^2,$$
where $\varepsilon > 0$.

$$\Rightarrow v_t = u_t = \gamma u_{xx} + F = \gamma v_{xx} - 2\gamma\varepsilon + F$$
$$\Rightarrow v_t = \gamma v_{xx} + \tilde{F},$$
where $\tilde{F} < 0$.

Hence, v obtains max at $t=0$, or $x=a$, $x=b$. Letting $\varepsilon \to 0$ we conclude the same for u.

Corollary - Suppose $u(t,x)$ solves the heat equation $u_t = \gamma u_{xx}$, with $\gamma > 0$, for $a<x<b$, $0<t<c$. Set

$$B = \{(0,x) : a \le x \le b\} \cup \{(t,a) : 0 \le t \le c\} \cup \{(t,b) : 0 \le t \le c\},$$

and let

$$M = \max\{u(t,x) : (t,x) \in B\}, \quad m = \min\{u(t,x) : (t,x) \in B\}.$$

Then

$$m \le u(t,x) \le M.$$

proof:

The upper bound follows from the maximum principle. Also by the maximum principle it follows that $-u(t,x) \le -m$ and thus $u(t,x) \ge m$.

Theorem - There is at most one solution to the following partial differential equation

$$u_t = \gamma u_{xx} + F(t,x)$$
$$u(0,x) = f(x)$$

proof

Let v, w solve the PDE and let $z = v - w$. Therefore,

$$z_t = \gamma z_{xx}$$
$$z(0,x) = 0$$

By the maximum principle it follows that

$$0 \le z(t,x) \le 0$$
$$\Rightarrow z = 0.$$

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Example:

If u, v are two solutions to

$$u_t = \gamma u_{xx}$$

such that $u \le v$ when $t=0$ or $x=a$ or $x=b$ then $u(t,x) \le v(t,x)$.

proof:

Let $z = v-u$. By the maximum principle

$$0 \le z \le \max\{v-u\}$$

$$\Rightarrow \quad 0 \le v-u \le \max\{v-u\}$$

$$\Rightarrow \quad u \le v$$

$\blacksquare$

Example:

Prove uniqueness of solutions to the following PDE:

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

on the interval $1<x<2$, with initial and boundary conditions

$$u(0,x) = f(x)$$

$$u(t,1) = \alpha(t)$$

$$u(t,2) = \beta(t)$$

proof:

1. We first establish a maximum principle. Consider the following rectangular domain:

$$R = \{0 \le t \le c \text{ and } 1 \le x \le 2\}$$

Let $v = u + \varepsilon x^2$. Then,

$$v_t = u_t, \quad v_x = u_x + 2\varepsilon x, \quad v_{xx} = u_{xx} + 2\varepsilon$$

$$\Rightarrow v_t = x u_{xx} + u_x$$

$$= x v_{xx} - 2\varepsilon x + v_x - 2\varepsilon x$$

$$\Rightarrow v_t = x v_{xx} + v_x - 4\varepsilon x.$$

Let (t^*, x^*) denote a maximum of v on the interior of R. It follows that

$$v_t(t^*,x^*) = v_x(t^*,x^*) = 0 \quad \text{and} \quad v_{xx}(t^*,x^*) \le 0.$$

$$\Rightarrow \quad 0 \le -4\varepsilon x^*$$

which is a contradiction. Therefore, the maximum occurs on the boundary of R. The result follows as $\varepsilon \to 0$.

The maximum cannot occur at $t=c$ since as before $u_t(c, x^*) \geq 0$, and $u_{xx}(c, x^*) \leq 0$.

2. To prove uniqueness we note that this PDE also satisfies a minimum principle since $-u$ is also a solution. Uniquess thus follows from the usual argument.