Lecture 2: Linear Transport Equation

Boring PDEs

Example:

What is the general solution of

$$u_t = x?$$

We can integrate with respect to t:

$$u(t,x) = xt + f(x),$$

where $f(x) \in C^1$ is a generic function.

Example:

Solve the following:

$$u_t + u^2 = 0$$

$$u(0,x) = f(x) \quad \text{(Initial Condition)}$$

We can seperate variables:

$$\frac{\partial u}{\partial t} = -u^2$$

$$\Rightarrow -\int_{f(x)}^{u} \frac{1}{u^2}\,du = \int_0^t dt$$

$$\Rightarrow u^{-1} - f(x)^{-1} = t$$

$$\frac{1}{u} - \frac{1}{f(x)} = t$$

$$\Rightarrow \frac{1}{u} = t + \frac{1}{f(x)}$$

$$\Rightarrow u = \frac{f(x)}{f(x)t + 1}$$

For a fixed value of x, this function blows up at a critical time

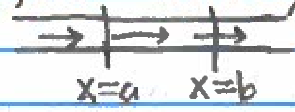
$$\tau(x) = -\frac{1}{f(x)}$$

If $f(x) > 0$ then, $\lim_{t\to\infty} u(t,x) = 0$.

Transport Equation

Conservation Laws:

$u(t,x)$ – density of some physical quantity.



$$\underbrace{\frac{d}{dt}\int_a^b u(t,x)\,dx}_{\substack{\text{total amount}\\ \text{of stuff in}\\ \text{interval } [a,b]}} = \underbrace{-F(u)\Big|_a^b}_{\substack{\text{rule for flux.}\\ \text{Comes from modeling/physics.}}} \quad \text{(Integral Form)}$$

$$\Rightarrow \int_a^b u_t(t,x)\,dx = -\int_a^b \frac{d}{dx}F(u(t,x))\,dx$$

$$\Rightarrow \int_a^b u_t(t,x)\,dx = -\int_a^b F'(u(t,x))\,u_x(t,x)\,dx$$

$$\Rightarrow \int_a^b u_t(t,x)\,dx + \int_a^b F'(u(t,x))\,u_x(t,x)\,dx = 0$$

As $b-a \to 0$ we obtain the following PDE:

$$u_t + F'(u)\,u_x = 0. \quad \text{(Differential Form)}$$

Example:

$$u_t + c\,u_x = 0 \qquad (F(u) = cu)$$

$$u(0,x) = f(x)$$

Assume $f(x)$ satisfies $\lim_{|x|\to\infty} f(x) = 0$ and $\lim_{|x|\to\infty} u(t,x) = 0$.

Total density is conserved since:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(t,x)\,dx = \int_{-\infty}^{\infty} u_t(t,x)\,dx$$

$$= -c\int_{-\infty}^{\infty} u_x(t,x)\,dx$$

$$= -c\,(u(\infty) - u(-\infty))$$

$$= 0$$

Change variables:

$z = x - ct$ (Travelling frame at speed c)

$\tau = t$

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -c\frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}$$

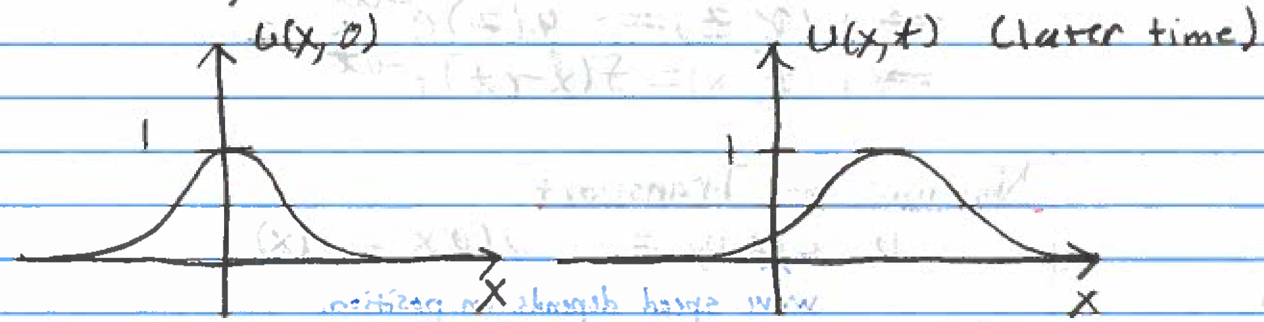
$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}$$

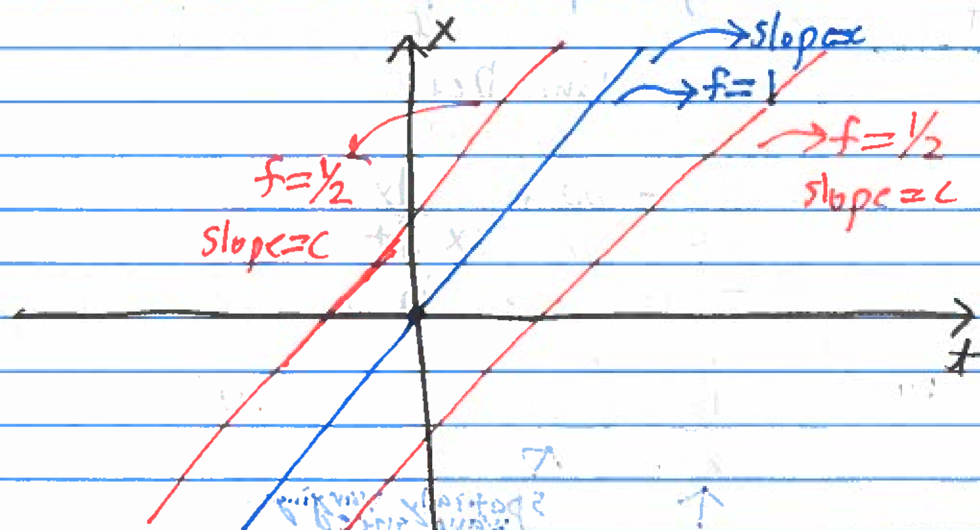
$$\Rightarrow U_t + cU_x = U_\tau = 0$$

$$\Rightarrow U = g(z)$$

To satisfy initial conditions

$u(t,x) = f(x-ct)$.





Since $f(0)=1$, it follows that along the curve $x=ct$, $u(t,x)=1$.

The lines $x=ct+b$ are characteristic curves. On the characteristic curves u is constant.

Damped Transport Equation

Solve,

$$u_t + c u_x + a u = 0, \quad u(0,x) = f(x)$$

Let

$$z = x - ct$$
$$\tau = t$$

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$$\Longrightarrow u_\tau + a u = 0$$

$$\Longrightarrow u_\tau = -au$$

$$\Longrightarrow u(\tau, z) = g(z)e^{-at}$$

$$\Longrightarrow u(t,x) = f(x-ct)e^{-at}$$

Nonuniform Transport

$$u_t + c(x) u_x = 0, \quad u(0,x) = f(x)$$

wave speed depends on position.

Idea: Track the value of $u(t,x)$ along a curve $(t, x(t))$ in the t-x plane. Define

$$h(t) = u(t, x(t))$$

$$\Longrightarrow \frac{dh}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

If $\frac{dx}{dt} = c(x)$, then u is constant along the solution to the equation

$$\frac{dx}{dt} = c(x).$$

velocity in moving frame

spatially varying wave speed

Definition - The graph of a solution $x(t)$ to

$$\frac{dx}{dt} = c(x)$$

is a characteristic curve for $u_t + c(x)u_x = 0$.

Example:

$$u_t + x u_x = 0 \qquad u(0,x) = f(x)$$

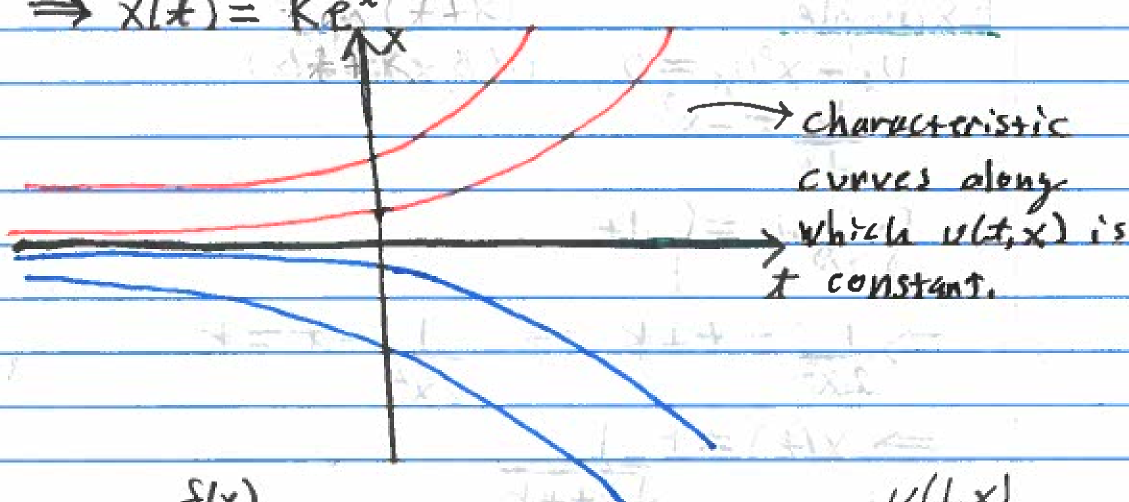
The characteristic curves are given by:

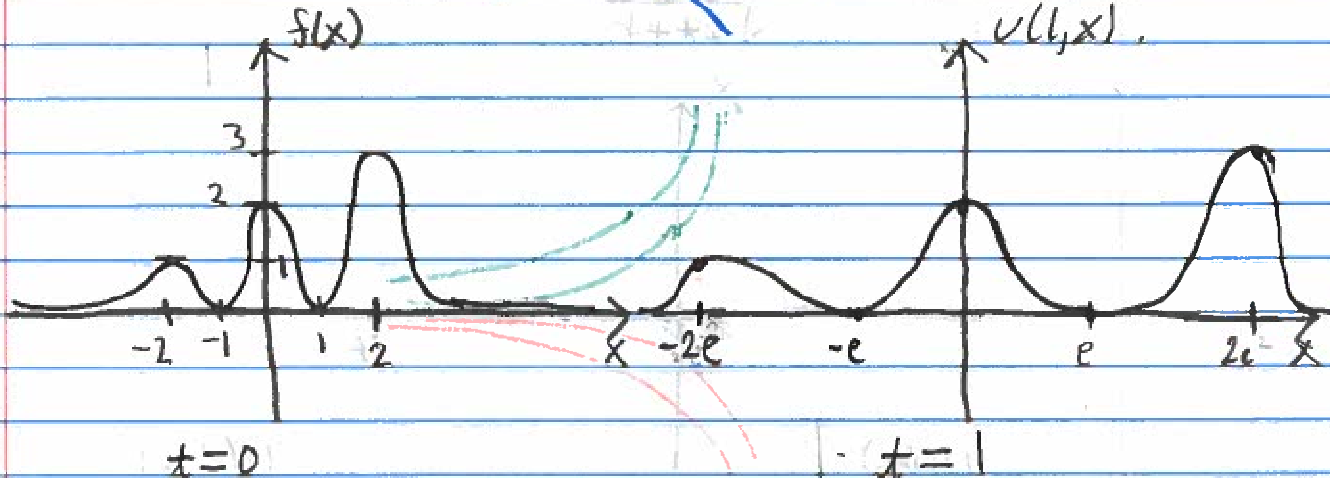
$$\frac{dx}{dt} = x$$

$$\int \frac{1}{x}\, dx = \int dt$$

$$\Rightarrow \ln(|x|) = t + k$$

$$\Rightarrow x(t) = k e^{t}$$





Suppose we wanted to find a general formula. Let $z = \ln(|x|) - t$. It follows that $u(t,x)$ is constant along the curves $\ln(|x|) = t + k$ and thus

$$u(t,x) = h(z), \text{ i.e. a function of } z.$$

Therefore,

$$u(0,x) = f(x)$$

$$\Rightarrow h(\ln(|x|)) = f(x)$$

$$\Rightarrow h(x) = f(e^{x})$$

$$\Rightarrow u(t,x) = h(z) = f(e^{z})$$

However, $z = \ln(x) - t$

$$\Rightarrow u(t,x) = f\left(e^{\ln(x)-t}\right)$$
$$= f(xe^{-t})$$

Check:

$$u_t = -xe^{-t} f'(xe^{-t})$$
$$u_x = e^{-t} f'(xe^{-t})$$

Example:

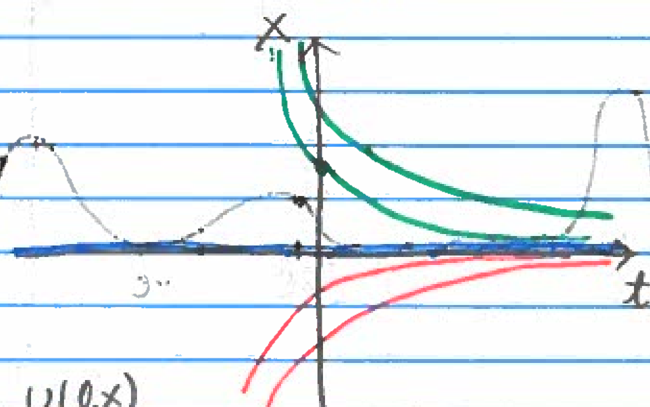
$$u_t - x^3 u_x = 0, \quad u(0,x) = f(x)$$

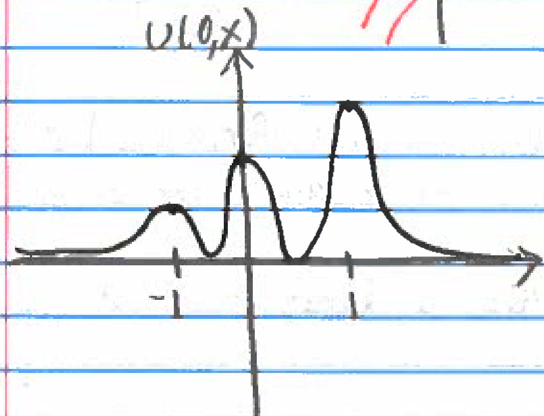
$$\frac{dx}{dt} = -x^3$$

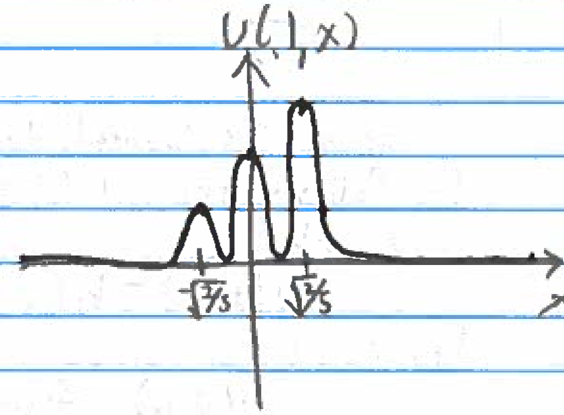
$$\int \frac{-1}{x^3}\,dx = \int dt$$

$$\Rightarrow \frac{1}{2x^2} = t + k \Rightarrow \frac{1}{2x^2} - t = k$$

$$\Rightarrow x(t) = \pm \frac{1}{\sqrt{2t+k}}$$







$t=0$ $\qquad$ $t=1$