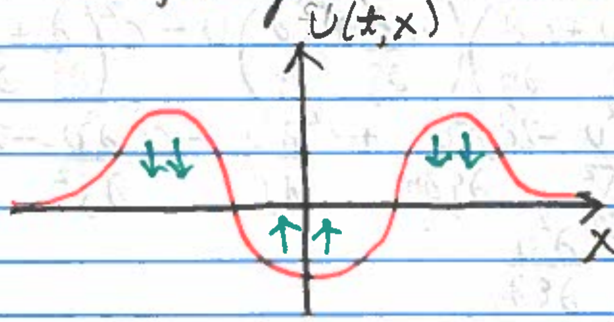
Lecture 3: The Wave Equation - d'Alembert's Formula

$$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{accleration}} = c^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{curvature}}, \quad u(t,x) - \text{Displacement of medium.}$$

$u(0,x) = f(x)$ (Initial Displacement)

$u_t(0,x) = g(x)$ (Initial Velocity)



↓↑ Direction of accleration.

d'Alemberts Solution

Derivation 1:

The wave operator $\square$ is defined by

$$\square u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2},$$

which can be factored as

$$\square u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

$$= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$$

$\Rightarrow$ If $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$ or $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$,

~~Then~~ then $\square u = 0$.

Consequently, by Lecture 2 and linearity it follows that

$$u(t,x) = p(x-ct) + q(x+ct)$$

is a solution.

$$\underbrace{p(x-ct)}_{\text{right travelling wave}}, \qquad \underbrace{q(x+ct)}_{\text{left travelling wave}}$$

Derivation 2:

Let $\xi = x - ct$, $\eta = x + ct$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}$$

$$= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \qquad\qquad = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$$

Therefore,

$$\Box u = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)u - c^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)u$$

$$= c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2} - c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} - c^2\frac{\partial^2 u}{\partial \eta^2}$$

$$= -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

Consequently,

$$\Box u = 0 \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

Therefore,

$$\frac{\partial u}{\partial \xi} = r(\xi)$$

$$\Rightarrow u = \int r(\xi)\,d\xi + q(\eta)$$

$$\Rightarrow u = p(\xi) + q(\eta)$$

$$\Rightarrow u = p(x-ct) + q(x+ct).$$

Initial Value Problem:

Return to the equation

$$u_{tt} = c^2 u_{xx}$$

$$u(0,x) = f(x)$$

$$u_t(0,x) = g(x)$$

We know $u(t,x) = p(x-ct) + q(x+ct)$.

$$\Rightarrow p(x) + q(x) = f(x)$$

$$-cp'(x) + cq'(x) = g(x)$$

$$\Rightarrow \begin{array}{l} cp'(x)+cq'(x)=cf'(x) \\ -cp'(x)+cq'(x)=g(x) \end{array}$$

$$\Rightarrow q'(x)=\frac{1}{2}f'(x)+\frac{g(x)}{2c}$$

$$\Rightarrow q(x)=\frac{1}{2}f(x)+\int_0^x \frac{g(z)}{2c}\,dz$$

$$p(x)=\frac{1}{2}f(x)-\int_0^x \frac{g(z)}{2c}\,dz$$

Therefore,

$$\boxed{u(t,x)=\frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz.}$$

Example:

$$u_{tt}=c^2u_{xx}$$

$$u(0,x)=e^{-x^2}+2e^{-(x-1)^2}$$

$$u_t(0,x)=0$$

$$\Rightarrow u(t,x)=\frac{1}{2}e^{-(x-ct)^2}+e^{-(x-c(t-1))^2}+\frac{1}{2}e^{-(x+ct)^2}+e^{-(x+c(t-1))^2}$$

(Solution Plotted in Mathematica)

Example:

$$u_{tt}=c^2u_{xx}$$

$$u(0,x)=0$$

$$u_t(0,x)=e^{-x^2}-2e^{-(x-1)^2}$$

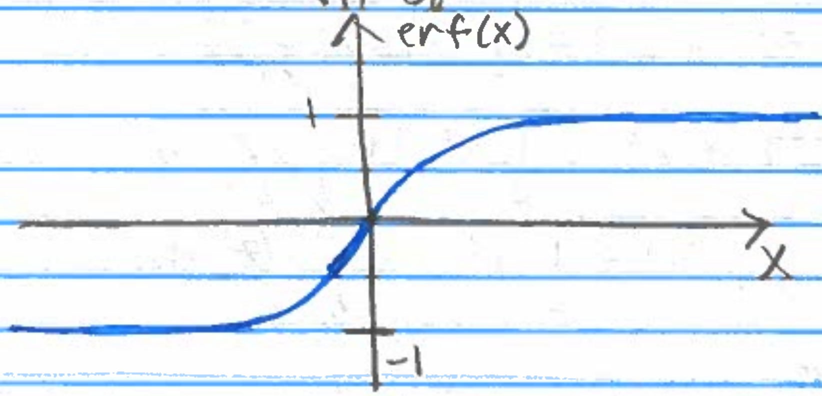
$$\Rightarrow u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct}\left(e^{-z^2}-2e^{-(z-1)^2}\right)dz$$

$$=\frac{\sqrt{\pi}}{4c}\left(\text{erf}(x+ct)-\text{erf}(x-ct)\right)$$

$$+\frac{\sqrt{\pi}}{2c}\left(\text{erf}(x+c(t-1))-\text{erf}(x-c(t-1))\right)$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\,dx$$



Plot of $u(t,x)$ in Mathematica.

<u>Uniqueness of Solutions.</u>

Let $u(t,x)$ solve

$$u_{tt} = c^2 u_{xx}$$
$$* \quad u(0,x) = f(x)$$
$$u_t(0,x) = g(x)$$

Define the energy $E(t)$ by

$$E(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right) dx$$

$$\Rightarrow \frac{dE}{dt} = \int_{\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{tx})\,dx$$

$$= \int_{\infty}^{\infty} (u_t u_{tt} - c^2 u_{xx} u_t)\,dx$$

$$= \int_{\infty}^{\infty} (u_t (c^2 u_{xx} - c^2 u_{xx}))\,dx$$

$$= 0.$$

Therefore, $E(t)$ is a constant given by:

$$E(t) = E(0) = \int_{-\infty}^{\infty} \left(\frac{1}{2} g(x)^2 + \frac{c^2}{2} f'(x)^2 \right) dx$$

Now, suppose u_1, u_2 solve *. Then, if we let $v = u_1 - u_2$ then

$$v_{tt} = c^2 v_{xx}$$
$$v(0,x) = 0$$
$$v_t(0,x) = 0.$$

With associated energy $E = 0$. Consequently, $v \equiv 0$ which implies $u_1 = u_2$.