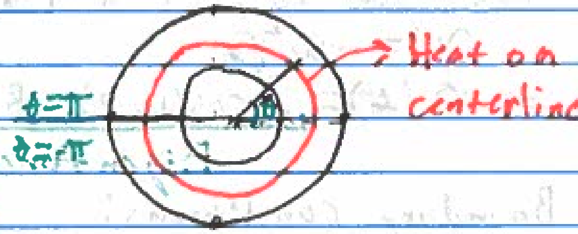
Lecture 4: Heated Ring

$$* \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \theta^2} \\ u(0,\theta) = f(\theta) \\ u(t,\pi) = u(t,-\pi) \\ u_t(t,\pi) = u_t(t,-\pi) \end{cases}$$

Initial conditions $\rightarrow u(0,\theta)=f(\theta)$

$u(t,\pi)=u(t,-\pi)$, $u_t(t,\pi)=u_t(t,-\pi)$ } Boundary Conditions



How do we solve:

Guess: $u(t,\theta) = T(t)\Theta(\theta)$

$$\Rightarrow T'(t)\Theta(\theta) = T(t)\Theta''(\theta)$$

$$\Rightarrow \frac{\Theta''}{\Theta} = \frac{T'}{T} = \lambda \longrightarrow$$ Must be a constant.

$$\Rightarrow * \Theta'' = \lambda\Theta, \quad T' = \lambda T *$$

$$\Rightarrow \Theta(\theta) = \underbrace{Ae^{\sqrt{\lambda}\theta} + Be^{-\sqrt{\lambda}\theta}}_{\text{Eigenfunction of } \frac{d^2}{d\theta^2}} \quad (\text{If } \lambda \neq 0)$$

Case 1 ($\lambda > 0$):

Boundary Conditions Imply

$$\Theta(\pi) = \Theta(-\pi) \Rightarrow Ae^{\sqrt{\lambda}\pi} + Be^{-\sqrt{\lambda}\pi} = Ae^{-\sqrt{\lambda}\pi} + Be^{\sqrt{\lambda}\pi}$$

$$\Theta'(\pi) = \Theta'(-\pi) \Rightarrow \sqrt{\lambda}Ae^{\sqrt{\lambda}\pi} + \sqrt{\lambda}Be^{-\sqrt{\lambda}\pi} = \sqrt{\lambda}Ae^{-\sqrt{\lambda}\pi} - \sqrt{\lambda}Be^{\sqrt{\lambda}\pi}$$

$\Rightarrow$ This is only possible if $A = B = 0$.

Case 2 ($\lambda = 0$):

$$\Theta'' = 0$$

$$\Rightarrow \Theta = A\theta + b$$

Boundary conditions imply $A = 0$. Therefore,

$$u(t,\theta) = b$$

Solves the P.D.E and boundary conditions but not the initial conditions.

Case 3 ($\lambda<0$):

Let $\lambda=-\omega^2$, It follows that:

$$\Theta''=-\omega^2\Theta$$

$$\Rightarrow \Theta(\theta)=\underbrace{A\cos(\omega\theta)+B\sin(\omega\theta)}_{\text{Eigen functions of } \frac{d^2}{d\theta^2}}.$$

Boundary conditions:

$$A\cos(\omega\pi)+B\sin(\omega\pi)=A\cos(\omega\pi)-B\sin(\omega\pi)$$

$$-A\omega\sin(\omega\pi)+B\omega\cos(\omega\pi)=A\omega\sin(\omega\pi)+B\omega\cos(\omega\pi)$$

$$\Rightarrow B\sin(\omega\pi)=0$$

$$A\sin(\omega\pi)=0$$

If $\omega\neq n\in\mathbb{N}$, $A=B=0$

If $\omega=n\in\mathbb{N}$, A, B are free variables.

Therefore, for $n\in\mathbb{N}$,

$$u(t,\theta)=e^{-n^2t}(\cos(n\theta)+\sin(n\theta))$$

solves the P.D.E. but not the initial conditions.

General Solution:

Take an infinite linear combination:

$$\boxed{u(t,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_ne^{-n^2t}\cos(n\theta)+\sum_{n=1}^{\infty}b_ne^{-n^2t}\sin(n\theta).}$$

Initial Conditions imply:

$$u(0,\theta)=f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\theta)+\sum_{n=1}^{\infty}b_n\sin(n\theta).$$

a_0:

$$\int_{-\pi}^{\pi}f(\theta)\,d\theta=\frac{1}{2}\int_{-\pi}^{\pi}a_0\,d\theta+\sum_{n=1}^{\infty}a_n\cancelto{0}{\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta}$$

$$+\sum_{n=1}^{\infty}b_n\cancelto{0}{\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta}$$

$$\Rightarrow \boxed{a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta.}$$

a_n:

$$\int_{-\pi}^{\pi} \cos(m\theta) f(\theta)\,d\theta = \underbrace{\frac{1}{2}\int_{-\pi}^{\pi} a_0 \cos(m\theta)}_{\to 0} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos(m\theta)\cos(n\theta)\,d\theta$$
$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin(n\theta)\cos(m\theta)\,d\theta.$$

Now,

$$\cos(m\theta)\cos(n\theta) = \frac{1}{2}\cos((m-n)\theta) + \frac{1}{2}\cos((m+n)\theta)$$
$$\sin(n\theta)\cos(m\theta) = \frac{1}{2}\sin((n+m)\theta) + \frac{1}{2}\sin((n-m)\theta)$$

$$\Rightarrow \int_{-\pi}^{\pi} a_n \cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} b_n \cos(n\theta)\sin(n\theta)\,d\theta = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos(m\theta) f(\theta)\,d\theta = \pi a_m$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta}$$

b_n:

$$\int_{-\pi}^{\pi} \sin(m\theta) f(\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} a_0 \sin(m\theta) + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos(n\theta)\sin(m\theta)\,d\theta$$
$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin(n\theta)\sin(m\theta)\,d\theta$$
$$= \pi b_m$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta.}$$

Example:

Solve $u_t = u_{xx}$

$u(0,x) = \cos^2(x)$

$u(t,\pi) = u(t,-\pi)$

$u_x(t,\pi) = u_x(t,-\pi)$

A solution is

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2x)}{2}\,dx$$

$$= 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \boxed{\cos^2(x)\cos(nx)\,dx}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2x)}{2}\cdot\cos(nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2x)\cos(nx)\,dx$$

$$\Rightarrow a_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(2x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos(4x)}{2}\,dx$$

$$= 1/2$$

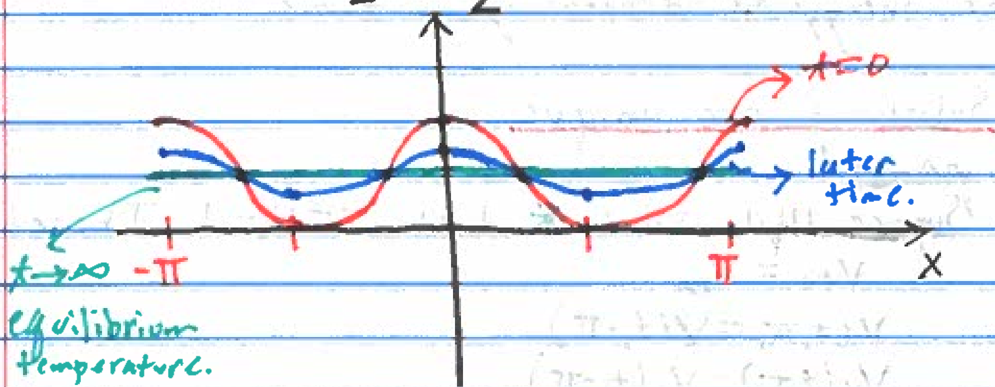
$a_n = 0$ if $n \neq 2$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\sin(nx)\,dx$$

$$= 0.$$

Therefore,

$$u(t,x) = \frac{1}{2} + \frac{e^{-4t}}{2}\cos(2x)$$



Properties:

1. $H(t) = \int_{-\pi}^{\pi} u(t,x)\,dx$

$$\frac{dH}{dt} = \int_{-\pi}^{\pi} u_t(t,x)\,dx$$

$$= \int_{-\pi}^{\pi} u_{xx}(t,x)\,dx$$

$$= u_x(t,x)\Big|_{-\pi}^{\pi}$$

$$= 0.$$

The total heat is conserved and is given by

$$H(t) = \int_{-\pi}^{\pi} u(0,x)\,dx$$

$$= \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \pi a_0.$$

2. Let $E(t) = \frac{1}{2}\int_{-\pi}^{\pi} u^2(t,x)\,dx$

$$\Rightarrow \frac{dE}{dt} = \int_{-\pi}^{\pi} u u_t \,dx$$

$$= \int_{-\pi}^{\pi} u u_{xx}\,dx$$

$$= \cancel{u u_x\Big|_{-\pi}^{\pi}}^{\,0} - \int_{-\pi}^{\pi} u_x^2\,dx$$

$$\Rightarrow \frac{dE}{dt} = -\int_{-\pi}^{\pi} u_x^2 \, dx \le 0.$$

The energy is decreasing in time.

3. Solutions are unique

proof:

Suppose u_1, u_2 solve $*$. Let $v = u_1 - u_2$. Therefore,

$$v_{tt} = v_{xx}$$
$$v(t,\pi) = v(t,-\pi)$$
$$v_x(t,\pi) = v_x(t,-\pi)$$
$$v(0,x) = 0$$

We know, $\frac{dE}{dt} \le 0$. However,

$$E(0) = \int_{-\pi}^{\pi} v(0,x)\,dx = \int_{-\pi}^{\pi} 0\,dx = 0.$$

Consequently, for all t, $E(t) = 0$. Therefore, $v(t,x) = 0$ and thus $u_1 = u_2$.

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