Lecture 5: Fourier Series

L^2-Inner Product

We want to write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

* The operation $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V if:
  1. $\langle a\vec{v}, \vec{w} \rangle = a \langle \vec{v}, \vec{w} \rangle$
  2. $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$
  3. $\langle \vec{v}, \vec{w} + \vec{z} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{v}, \vec{z} \rangle$

The L^2 inner product on $[-\pi, \pi]$ is defined by:

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$$\Rightarrow \|f\| = \langle f, f \rangle^{1/2} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx}.$$

Orthogonality:
1. $\langle \cos(mx), \cos(nx) \rangle = 0$ if $m \neq n$.
2. $\langle \sin(mx), \sin(nx) \rangle = 0$ if $m \neq n$.
3. $\langle \cos(mx), \sin(nx) \rangle = 0$
4. $\langle \cos(mx), \cos(mx) \rangle = 1$
5. $\langle \sin(mx), \sin(mx) \rangle = 1$
6. $\langle 1, 1 \rangle = 2$

$\Rightarrow$ The trigonometric functions form an orthogonal system. This is very similar to a basis.

Example:

$f(x) = x$

$$\Rightarrow x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Using orthogonality:

1. $\langle x, 1 \rangle = \frac{a_0}{2} \langle 1, 1 \rangle = a_0$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0.$$

2. $\langle x, \cos(mx) \rangle = a_m \langle \cos(mx), \cos(mx) \rangle$

$$\Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(mx) \, dx = 0.$$

3. $\langle x, \sin(mx) \rangle = b_m \langle \sin(mx), \sin(mx) \rangle$

$$\Rightarrow b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(mx) \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin(mx) \, dx$$

$$= \frac{2}{\pi} \left[-\frac{x \cos(mx)}{m} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos(mx)}{m} \, dx \right]$$

$$= \frac{2}{\pi} \; \frac{-\pi \cos(m\pi)}{m}$$

$$= \frac{2 \cdot (-1)^{m+1}}{m}$$

Therefore,

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{m} \sin(nx)$$

$$= 2 \left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots \right)$$

Convergence Issues:

$$f(x) = x \sim 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right)$$

1. $f(0) = 0 = 2\left(\sin(0) - \frac{\sin(2\cdot 0)}{2} + \frac{\sin(3\cdot 0)}{3} + \dots\right)$ ✓

2. $f(\pi/2) = 2\left(\sin(\pi/2) - \frac{\sin(\pi)}{2} + \frac{\sin(3\cdot\pi/2)}{3} + \dots\right)$

$$= 2\left(1 - 1/3 + 1/5 + \dots\right)$$
$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$$
$$= \pi/2$$ ✓

3. $f(\pi) = \pi = 2\cdot\left(\sin(\pi) - \frac{\sin(2\cdot\pi)}{2} + \frac{\sin(3\cdot\pi)}{3} + \dots\right)$

$\Rightarrow \pi = 0$ ☹

4. $f(3\pi/2) = 2\left(\sin\left(\frac{3\pi}{2}\right) - \frac{\sin(3\pi)}{2} + \frac{\sin(9\pi/2)}{3} + \dots\right)$

$$= 2\left(-1 + \frac{1}{3} - \frac{1}{5} + \dots\right)$$
$$= -2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$$
$$= -\pi/2$$

$\Rightarrow \frac{3\pi}{2} = -\pi/2$ ☹

* The issue is that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

is a 2π-periodic function while $f(x)=x$ is not.

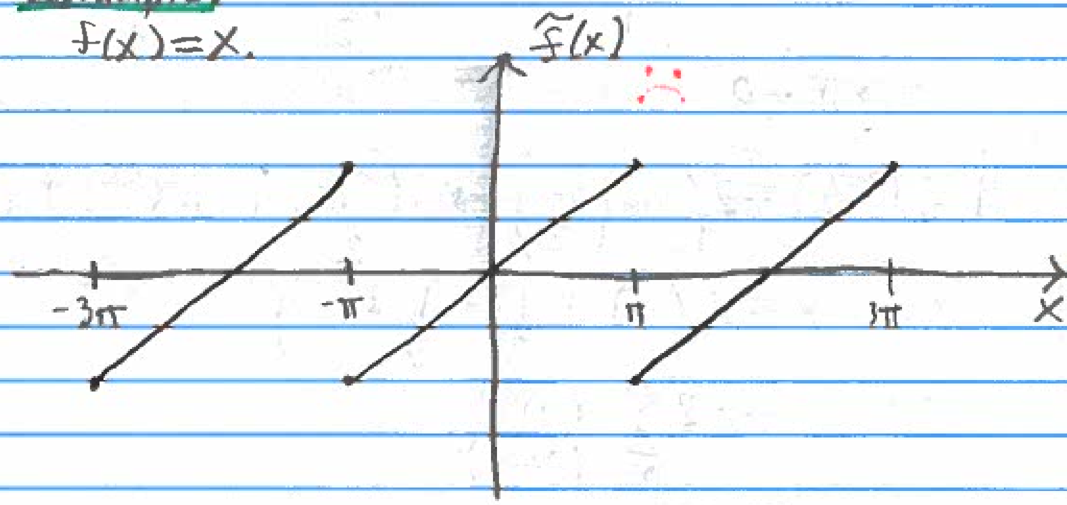
Periodic Extensions:

Lemma - If $f(x)$ is any function defined for $-\pi < x < \pi$, then there is a unique 2π-periodic function $\tilde{f}$, known as the 2π periodic extension of f that satifies $\tilde{f}(x) = f(x)$ for all $-\pi < x < \pi$.

proof:
Let $x \in \mathbb{R}$. There exists $m \in \mathbb{Z}$ such that $(2m-1)\pi < x \leq (2m+1)\pi$. Define,

$$\tilde{f}(x) = f(x - 2m\pi).$$

Example:
$f(x) = x$.



Theorem - If $\tilde{f}(x)$ is a 2π-periodic, piecewise C^1 function then at any x:

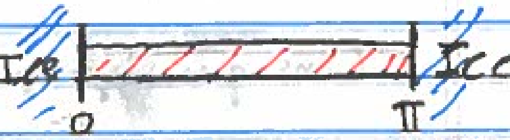
1. $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$, (if $\tilde{f}$ is continuous at x)

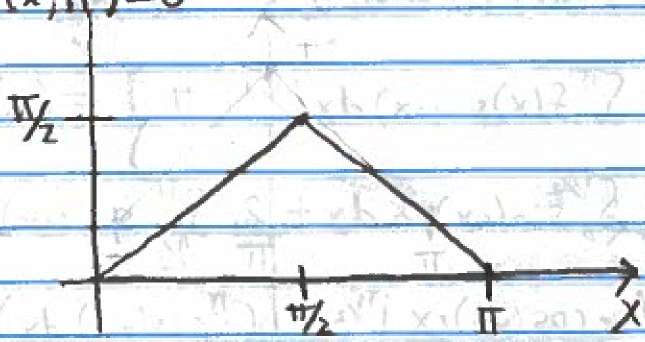
2. $\frac{1}{2}[\tilde{f}(x^+) + \tilde{f}(x^-)]$
$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$, (if $\tilde{f}$ has a jump at x)

- Brittany

Cooling of a Rod:

$$u_t = u_{xx}$$
$$u(0,x) = f(x) = \pi/2 - |x - \pi/2|$$
$$u(t,0) = 0$$
$$u(t,\pi) = 0$$





Guess:

$$u(t,x) = T \cdot X$$

$$\Rightarrow T'X = TX''$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda$$

The $\lambda < 0$ eigenfunctions are the only ones that yield nonzero eigenfunctions.

Let $\lambda = -w^2$. Therefore,

$$X = A\cos(wx) + B\sin(wx), \quad T = e^{-w^2 t}$$

$$u(t,0) = 0 \Rightarrow A = 0$$
$$u(t,\pi) = 0 \Rightarrow B\sin(w\pi) = 0$$

Therefore, to have nontrivial solutions we need

$$w = n \in \mathbb{N}.$$

The generic solution is given by:

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

$$\Rightarrow u(0,x) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Consequently, by orthogonality:

$$\int_0^{\pi} f(x)\sin(nx)\,dx = b_n \int_0^{\pi} \sin^2(nx)\,dx$$

$$\Rightarrow \frac{\pi}{2} b_n = \int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi/2} \sin(nx)\cdot x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left(-\frac{\cos(nx)\cdot x}{n}\Bigg|_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos(nx)}{n}\,dx \right)$$

$$+ \frac{2}{\pi}\left(-\frac{\cos(nx)(\pi - x)}{n}\Bigg|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\cos(nx)}{n}\,dx \right)$$

$$= -\frac{2}{\pi}\left(\frac{\cos(n\pi/2)}{n} + \frac{\sin(nx)}{n^2}\Bigg|_{\pi/2}^{\pi} \right) + \frac{2}{\pi n}\cos(n\pi/2)$$

$$- \frac{2}{\pi}\,\frac{\sin(nx)}{n^2}\Bigg|_{\pi/2}^{\pi}$$

$$= \frac{4}{\pi}\,\frac{\sin(n\pi/2)}{n^2}$$

$$\Rightarrow b_n = \frac{4}{\pi}\begin{cases} 1/n^2 & \text{if } n = 1, 5, 9, \ldots \\ -1/n^2 & \text{if } n = 3, 7, 11, \ldots \\ 0 & \text{if } n = \text{even} \end{cases}$$

$$\Rightarrow \boxed{u(t,x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-(2n-1)^2 t} \sin((2n-1)x)}$$

Convergence and Regularity

Let I be an interval

1. $f_n(x)$ converges to $f(x)$ pointwise if for all $x \in I$
$$\lim_{n\to\infty} f_n(x) = f(x).$$

2. $f_n(x)$ converges to $f(x)$ uniformly if for all $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that
$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in I \text{ and } n \geq N$$

$$\Rightarrow \lim_{n\to\infty} \max_{x\in I} |f_n(x) - f(x)| = 0$$

3. $f_n(x)$ converges to $f(x)$ in L^2 if
$$\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0$$

Theorem - Suppose $f \in L^2$ on $[a,b]$ with Fourier series
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

Let
$$f_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx).$$

1. $\lim_{m\to\infty} \|f_m - f\|_{L^2} = 0$

2. If the periodic extension of f is continuous then f_m converges uniformly to f.

Theorem - If the periodic extension of f is in C^k, then

$$|a_n| < \frac{C}{n^{k+1}}, \quad |b_n| < \frac{C}{n^{k+1}}$$

proof:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x)\,dx$$

$$= \cancel{\frac{1}{n\pi} \sin(nx) f(x)\Big|_{-\pi}^{\pi}}^{\,0} - \frac{1}{n\pi}\int_{-\pi}^{\pi} \sin(nx) f'(x)\,dx$$

$$= \frac{1}{n^2\pi} \cos(nx) f'(x)\Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} \cos(nx) f''(x).$$

The result follows by induction

Bessel's Equality -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi} \frac{a_0^2}{4} dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n^2 \cos^2(nx) + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n^2 \sin^2(nx)\,dx$$

$$= \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \boxed{\|f\|_{L^2}^2 = \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$