

Lecture 6: Complex Numbers and Fourier Series

Example:

Find the Fourier series for

$$f(x) = \sin^5(x)$$

on $[-\pi, \pi]$. We can use Euler's formula to simplify.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$\begin{aligned}\sin^5\theta &= \frac{1}{(2i)^5} (e^{i\theta} - e^{-i\theta})^5 \\ &= \frac{1}{32i} (e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}) \\ &= \frac{1}{16} \sin(5\theta) - \frac{5}{16} \sin(3\theta) + \frac{5}{8} \sin(\theta)\end{aligned}$$

$$\Rightarrow b_1 = \frac{5}{8}, \quad b_3 = -\frac{5}{16}, \quad b_5 = \frac{1}{16}$$

Example:

Find the Fourier series for

$$f(x) = \sin(\sqrt{2}x)$$

on the interval $[0, 2]$.

The orthogonal system for $[0, 2\pi]$ is

$$\{1, \cos(x), \cos(2x), \dots, \sin(x), \sin(2x), \dots\}$$

We compress these functions by the factor $\frac{2}{2\pi} = \frac{1}{\pi}$.

Consequently, an orthogonal system on $[0, 2]$ is:

$$\{1, \cos(\pi x), \cos(2\pi x), \dots, \sin(\pi x), \sin(2\pi x), \dots\}$$

$$\Rightarrow \sin(\sqrt{2}x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Coefficients:

$$1. \int_0^2 \sin(\sqrt{2}x) dx = \int_0^2 \frac{a_0}{2} dx$$

$$\Rightarrow -\frac{\cos(\sqrt{2}x)}{\sqrt{2}} \Big|_0^2 = a_0$$

$$a_0 = \frac{1}{\sqrt{2}} (1 - \cos(2\sqrt{2}))$$

$$2. \int_0^2 \sin(\sqrt{2}x) \sin(n\pi x) dx = \int_0^2 b_n \sin^2(n\pi x) dx$$

$$\Rightarrow \int_0^2 \sin(\sqrt{2}x) \sin(n\pi x) dx = b_n$$

$$\Rightarrow \int_0^2 \frac{(e^{i\sqrt{2}x} - e^{-i\sqrt{2}x})}{2i} \frac{(e^{in\pi x} - e^{-in\pi x})}{2i} dx = b_n$$

$$\Rightarrow -\frac{1}{4} \int_0^2 (e^{i(\sqrt{2}+n\pi)x} - e^{i(\sqrt{2}-n\pi)x} - e^{-i(\sqrt{2}-n\pi)x} + e^{-i(\sqrt{2}+n\pi)x}) dx = b_n$$

$$\Rightarrow -\frac{1}{4} \int_0^2 (\cos((\sqrt{2}+n\pi)x) - \cos((\sqrt{2}-n\pi)x)) dx = b_n$$

$$\Rightarrow \frac{1}{2} \int_0^2 [\cos((\sqrt{2}-n\pi)x) - \cos((\sqrt{2}+n\pi)x)] dx = b_n$$

$$\Rightarrow \frac{1}{2} \left(\frac{\sin(2(\sqrt{2}-n\pi)) - 1}{\sqrt{2}-n\pi} - \frac{\cos(2(\sqrt{2}+n\pi)) + 1}{\sqrt{2}+n\pi} \right) = b_n$$

$$3. a_n = 0.$$

Differentiation and Integration

1. Given

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx),$$

What is the Fourier series of x^2 ?

$$\Rightarrow \frac{x^2}{2} \sim C + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+2}}{k^2} \cos(kx)$$

$$\Rightarrow x^2 \sim C + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)$$

Need to find c :

$$\int_{-\pi}^{\pi} x^2 dx = 2\pi c$$

$$\frac{2\pi^3}{3} = 2\pi c$$

$$\Rightarrow c = \frac{\pi^2}{3}$$

$$\Rightarrow x^2 \sim \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)$$

2. What is the Fourier series of x^3 ?

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)$$

$$\Rightarrow \frac{x^3}{3} \sim C + \frac{\pi^2}{3} x + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx)$$

$$\Rightarrow x^3 \sim C + \pi^2 x + 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx)$$

$$\sim C + \sum_{k=1}^{\infty} \left(\frac{2\pi^2 (-1)^{k+1}}{k} + \frac{12(-1)^k}{k^3} \right) \sin(kx)$$

Since $\int_{-\pi}^{\pi} x^3 dx = 0$ it follows that $c = 0$.

3. What about a Fourier series for $f(x) = x$. If we just differentiate we have:

$$1 \sim 2 \sum_{k=1}^{\infty} (-1)^{k+1} \cos(kx)$$

This is not even remotely true!!

* If $\tilde{f} \in C^1$ then the Fourier series for $f'(x)$ is the term by term derivative for the Fourier series of $f(x)$.