Lecture 7: Heat Equation

$$u_t = \gamma u_{xx}$$
$$u(0,x) = f(x)$$

Dirichlet Boundary Conditions

$$u(t,0) = u(t,\ell) = 0.$$

Properties:

1. Let $E(t) = \int_0^\ell u(t,x)^2 dx.$

$$\Rightarrow \frac{dE}{dt} = \int_0^\ell 2 u_t u \, dx$$
$$= \int_0^\ell 2 u_{xx} u \, dx$$
$$= \cancel{2 u_x u \Big|_0^\ell}^{\,0} - 2\int_0^\ell u_x^2 dx$$

$$\Rightarrow \frac{dE}{dt} \leq 0$$

Solutions are unique.

2. Look for seperable solutions

$$u(t,x) = T X$$
$$\Rightarrow T' = \lambda T, \quad X'' = \lambda X$$

Boundary conditions imply

$$X = b_n \sin\left(\frac{n\pi x}{\ell}\right), \quad T = \exp\left(\frac{-n^2\pi^2}{\ell^2} t\right)$$

Generic solution is of the form:

$$u(t,x) = \sum_{n=1}^{\infty} b_n \exp\left(\frac{-n^2\pi^2}{\ell^2} t\right) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow u(0,x) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = b_n \int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann Boundary Conditions.

- $u_t = u_{xx}$

$u_x(t,0) = 0$ $\Big\rangle$ Insulated Boundary.

$u_x(t,1) = 0$

$u(0,x) = -x(x-1) = -x^2 + x$

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x)$$

$$\Rightarrow u(0,x) = -x^2 + x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\Rightarrow a_0 = 2\int_0^1 (-x^2 + x)\,dx$$

$$= 1/3$$

$$a_n = 2\int_0^1 (-x^2+x)\cos(n\pi x)\,dx$$

$$= -\frac{2}{n^2\pi^2}\left(1 + (-1)^n\right)$$

$$= \frac{2}{n^2\pi^2}\left((-1)^{n+1} - 1\right)$$

Inhomogeneous Boundary Conditions

$$u_t = \gamma u_{xx}$$
$$u(t,0) = \alpha$$
$$u(t,l) = \beta.$$
$$u(0,x) = f(x)$$

Assume there exists a steady state solution:

$$\lim_{t\to\infty} u(t,x) = u^*(x)$$

$$\Rightarrow \quad u^*_{xx} = 0$$
$$\Rightarrow \quad u^*(x) = ax + b$$
$$u^*(0) = b = \alpha$$
$$u^*(l) = al + \alpha = \beta$$

$$\Rightarrow \quad u^*(x) = \frac{\beta - \alpha}{l} x + \alpha$$

Let $u(t,x) = v(t,x) + u^*(x)$

$$\Rightarrow \quad v_t = u_t$$
$$= \gamma u_{xx}$$
$$= \gamma(v_{xx} - \overset{0}{\cancel{u^*_{xx}}})$$

$$\Rightarrow \quad v_t = \gamma v_{xx}$$
$$v(t,0) = u(t,0) - u^*(0) = 0$$
$$v(t,l) = u(t,l) - u^*(l) = 0$$
$$v(0,x) = f(x) - u^*(x).$$

The PDE for $v(t,x)$ is given by:

$$v_t = \gamma v_{xx}$$
$$v(t,0) = 0$$
$$v(t,l) = 0$$
$$v(0,x) = f(x) - \frac{(\beta - \alpha)}{l} x - \alpha.$$