Lecture 8: Finite Difference Solutions to Heat Equation

Finite Difference Approximations

1. $u'(x) \approx \dfrac{u(x+h)-u(x)}{h}$

How accurate is this? Taylor expand with respect to h:

$$u(x+h) = u(x) + u'(x)\cdot h + \tfrac{1}{2}u''(\xi)h^2$$

$$\boxed{\Rightarrow u'(x) = \frac{u(x+h)-u(x)}{h} + \tfrac{1}{2}u''(\xi)h}$$

Therefore,

$$\boxed{\left|\frac{u(x+h)-u(x)}{h} - u'(x)\right| \leq C h} \qquad C = \max \tfrac{1}{2}|u''(\xi)|$$

The approximation is first order in h:

$$\boxed{u'(x) = \frac{u(x+h)-u(x)}{h} + O(h).}$$

2. To derive a second order approximation:

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + \dots$$
$$u(x-h) = u(x) - u'(x)h + \tfrac{1}{2}u''(x)h^2 - \tfrac{1}{6}u'''(x)h^3 + \dots$$

$$\Rightarrow u(x+h) - u(x-h) = 2u'(x)h + \tfrac{1}{3}u'''(x)h^3 + \dots$$

$$\Rightarrow \boxed{u'(x) = \frac{u(x+h)-u(x-h)}{2h} + O(h^2).}$$

3. Derive a second order forward operator:

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + \ldots$$

$$u(x+2h) = u(x) + 2u'(x)h + 2u''(x)h^2 + u'''(x)h^3 + \ldots$$

$$\Rightarrow 4u(x+h) - u(x+2h) = 3u(x) + 2u'(x)h - \tfrac{1}{3}u'''(x)h^3 + \ldots$$

$$\Rightarrow 4u(x+h) - u(x+2h) - 3u(x) = 2u'(x)h - \tfrac{1}{3}u'''(x)h^3 + \ldots$$

$$\Rightarrow \boxed{u'(x) = \frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} + \mathcal{O}(h^2)}$$

4. Second order backward operator

$$\boxed{u'(x) = \frac{u(x-2h) - 4u(x-h) + 3u(x)}{2h} + \mathcal{O}(h^2)}$$

5. Second order second derivative

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + \tfrac{1}{24}u''''(x)h^4$$

$$u(x-h) = u(x) - u'(x)h + \tfrac{1}{2}u''(x)h^2 - \tfrac{1}{6}u'''(x)h^3 - \tfrac{1}{24}u''''(x)h^4$$

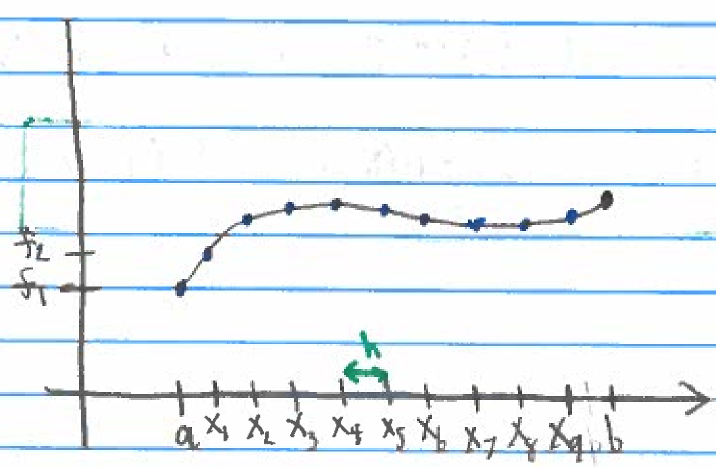
$$\Rightarrow u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + \mathcal{O}(h^4)$$

$$\Rightarrow \boxed{u''(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + \mathcal{O}(h^2)}$$

Matrix Representation

Discretize $f(x)$:

$$f_i = f(x_i), \quad x_i = a + ih$$



Define,

$$f_0' = \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h}.$$

$$f_n' = \frac{f_{n-2} - 4f_{n-1} + 3f_n}{2h}$$

Let $\vec{f}' = \begin{bmatrix} f_0' \\ f_1' \\ \vdots \\ f_n' \end{bmatrix}$, $\vec{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$. Then,

$$\boxed{\vec{f}' = D\vec{f}, \quad D = \frac{1}{2h}\begin{bmatrix} -3 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 3 \end{bmatrix}}$$

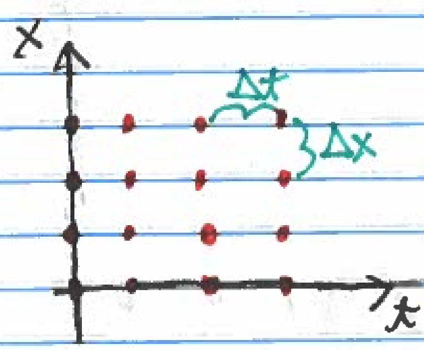
Note, if we want to approximate $f''(x)$ we can use:

$$\vec{f}'' = D^2\vec{f}.$$

Finite Difference Solution to PDE

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \ t > 0$$

Introduce a rectangular mesh of nodes $(t_i, x_j) \in \mathbb{R}^2$



$$\Delta t = t_{j+1} - t_j$$
$$\Delta x = x_{j+1} - x_j$$
$$u_{i,j} \approx u(t_i, x_j)$$

$$u_t(t_i, x_j) = \frac{u(t_{i+1}, x_j) - u(t_i, x_j)}{\Delta t} + O(\Delta t)$$

$$u_{xx}(t_i, x_j) = \frac{u(t_i, x_{j+1}) - 2u(t_i, x_j) + u(t_i, x_{j-1})}{(\Delta x)^2} + O(\Delta x^2).$$

Approximate solutions to the PDE satisfy:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \gamma \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta x^2} \right)$$

$$\Longrightarrow \boxed{u_{i+1,j} = u_{i,j} + \frac{\gamma \Delta t}{\Delta x^2} \left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \right)}$$

For stability we need

$$\frac{\gamma \Delta t}{\Delta x^2} < 1$$

$$\Longrightarrow \Delta t < \frac{1}{\gamma} \Delta x^2.$$