# Lecture 9: Stationary Solutions and Wave Equation

## Example:

What is the stationary solution to

$$u_t = u_{xx}$$
$$u(t,0) = \alpha \leftarrow \text{constant temperature}$$
☆ $$u_x(t,\ell) = \beta \leftarrow \text{constant flux.}$$
$$u(0,x) = f(x).$$

Stationary solution $u^*(x)$ satisfies

$$u^*_{xx} = 0$$
$$u^*(0) = \alpha$$
$$u^*_x(\ell) = \beta$$
$$\Rightarrow u^*(x) = ax + b$$
$$u^*(0) = \alpha \Rightarrow b.$$
$$u^*_x(0) = \beta = a$$

The stationary solution is given by:

$$u^*(x) = \beta x + \alpha$$

To solve ☆ assume a solution of the form

$$u(t,x) = v(t,x) + u^*(x).$$

$$\Rightarrow v_t + u^*_t(x) = v_{xx} + u^*_{xx}(x)$$
$$u(t,0) = v(t,0) + \alpha = \alpha$$
$$u_x(t,\ell) = v_x(t,\ell) + \beta = \beta$$
$$u(0,x) = v(0,x) + ax + b = f(x)$$

Therefore, we obtain the following system:

$$v_t = v_{xx}$$
☆☆ $$v(t,0) = 0$$
$$v_x(t,\ell) = 0$$
$$v(0,x) = f(x) - ax - b.$$

To solve ☆☆ we assume a seperable solution

$$u(t,x) = T \cdot X.$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -w^2$$

Boundary conditions require that

$$X = A\sin(wx).$$

$$\Rightarrow X'(l) = 0 \Rightarrow \cos(wl) = 0$$

$$\Rightarrow wl = \frac{(2n-1)\pi}{2}.$$

$$\Rightarrow w = \frac{(2n-1)\pi}{2l}$$

The solution can be written as

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2\pi^2}{4l^2}t} \sin\left(\left(\frac{2n-1}{2l}\right)\pi x\right)$$

$$b_n = \frac{2}{l}\int_0^l (f(x) - \beta x - \alpha)\sin\left(\left(\frac{2n-1}{2l}\right)\pi x\right) dx$$

Example:

What is the stationary solution of

$$u_t = u_{xx} + \sin(x)$$
$$u(t,0) = 0$$
$$u(t,1) = 0$$

Stationary solution satisfies:

$$0 = u^*_{xx} + \sin(x)$$

$$\Rightarrow u^*(x) = \sin(x) + ax + b$$

$$u^*(0) = 0 \Rightarrow b = 0$$

$$u^*(1) = 0 \Rightarrow \sin(1) + a = 0$$

$$\Rightarrow a = -\sin(1).$$

$$\Rightarrow u^*(x) = \sin(x) - \sin(1)x$$

Example:

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = f(x)$$
$$u_t(0,x) = 0$$
$$u(t,0) = 0$$
$$u(t,1) = 0$$

$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -w^2$$

$$\Rightarrow X = \sin(n\pi x), \quad T = b_n \sin(n\pi c t) + d_n \cos(n\pi c t).$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} \left(b_n \sin(n\pi c t) + d_n \cos(n\pi c t)\right) \sin(n\pi x).$$

Initial conditions imply $b_n = 0$ and
$$d_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$\Rightarrow u(t,x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} (-1)^{k+1} \frac{\cos((2n-1)\pi t)\sin((2n-1)\pi x)}{(2n+1)^2}.$$

Example:

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = 0$$
$$u_t(0,x) = f(x)$$
$$u_x(t,0) = 0$$
$$u_x(t,1) = 0$$

$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$$

$$u(t,x) = T \cdot X.$$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -w^2$$

General Solution:

$$X = A\cos(n\pi x), \text{ or } X = A.$$

$$\Rightarrow u(t,x) = a_0 + a_1 t + \sum_{n=1}^{\infty} \left(a_n \sin(cn\pi t) + b_n \cos(cn\pi t)\right)\cos(n\pi x)$$

Initial conditions imply:

$$a_0 = 0, \ b_n = 0.$$

$$u_t(0,x) = f(x) = a_1 + \sum_{n\ge 1}^{\infty} a_n cn\pi \cos(n\pi x)$$

$$\Rightarrow a_1 = \int_0^1 f(x)\,dx.$$

$$a_n = \frac{2}{c\pi}\int_0^1 f(x)\cos(n\pi x)\,dx.$$

$$= \frac{2}{c\pi} \cdot \frac{4\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)}{n^2\pi^2}$$