MST 205
Spring 2022
Exam #3
04/15/22

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	20	
2	15	
3	20	
4	20	
5	15	
6	10	
Total:	100	

Do not write in the table to the right.

1. (20 points) Consider the following differential equation:

$$\frac{dy}{dx} + 3x^2y^2 = 0.$$

(a) (5 points) **Short Answer:** For this differential equation, check all boxes that apply.

- [x] **First order**
- [ ] **Second order**
- [ ] **Linear**
- [x] **Nonlinear**
- [ ] **Constant coefficient**
- [x] **Non-constant coefficients**
- [x] **Homogeneous**
- [ ] **Non-homogeneous**

(b) (5 points) Show that $y = 0$ is a solution to this equation.

$$\frac{d0}{dx} + 3x^2(0) = 0$$

(c) (10 points) Find the general formula for other solutions to this equation.

$$\int \frac{1}{y^2}dy = -\int 3x^2 dx$$

$$\Rightarrow -\frac{1}{y} = -x^3 + C$$

$$\Rightarrow y = \frac{1}{C + x^3}.$$

2. (15 points) Consider the following differential equation:

$$\frac{dy}{dx} + 2xy = x.$$

(a) (5 points) **Short Answer:** For this differential equation, check all boxes that apply.

☑ **First order**	☐ **Second order**
☑ **Linear**	☐ **Nonlinear**
☐ **Constant coefficient**	☑ **Non-constant coefficients**
☐ **Homogeneous**	☑ **Non-homogeneous**

(b) (10 points) Find the general formula for solutions to this equation.

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$$

$$\Rightarrow e^{x^2}y = \frac{1}{2}e^{x^2} + c$$

$$\Rightarrow y = \frac{1}{2} + ce^{-x^2}.$$

3. (20 points) Consider the following differential equation:

$$y''(x) + y'(x) - 2y(x) = 0.$$

(a) (5 points) **Short Answer:** For this differential equation, check all boxes that apply.

- [ ] **First order**
- [x] **Second order**
- [x] **Linear**
- [ ] **Nonlinear**
- [x] **Constant coefficient**
- [ ] **Non-constant coefficients**
- [x] **Homogeneous**
- [ ] **Non-homogeneous**

(b) (10 points) Find the general solution to this equation.

$y(x) = e^{\lambda x}$

$\Rightarrow \lambda^2 + \lambda - 2 = 0$

$\Rightarrow (\lambda+2)(\lambda-1) = 0$

$\Rightarrow \lambda = -2, \lambda = 1$

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{x}$

(c) (5 points) Use your general to solve this differential equation with the following initial conditions

$$y(0) = 1,$$
$$y'(0) = 0.$$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow -2c_1 + c_2 = 0$

$c_2 = 2c_1$

$\Rightarrow 3c_1 = 1$

$\Rightarrow c_1 = 1/3$

$c_2 = 2/3$

$y = \frac{1}{3}e^{-2x} + \frac{2}{3}e^{x}.$

4. (20 points) Consider the following differential equation:

$$y''(x) + 2y(x) = 6e^{2x} - 4e^{-2x}.$$

(a) (5 points) **Short Answer:** For this differential equation, check all boxes that apply.

- [ ] **First order**
- [x] **Second order**
- [x] **Linear**
- [ ] **Nonlinear**
- [x] **Constant coefficient**
- [ ] **Non-constant coefficients**
- [ ] **Homogeneous**
- [x] **Non-homogeneous**

(b) (15 points) Find the general solution to this equation.

$$y_H = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

$$y_p = Ae^{2x} + Be^{-2x}$$

$$y_p'' = 4Ae^{2x} + 4Be^{-2x}$$

$$\Rightarrow 6A = 6, \quad 6B = -4$$

$$\Rightarrow A = 1, \quad B = -2/3$$

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + e^{2x} - 2/3\, e^{-2x}$$

5. (15 points) Suppose $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a linear transformation so that

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} \text{ and } T\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}.$$

(a) (5 points) Find

$$T\begin{bmatrix}1\\0\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix}.$$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}T\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}T\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}T\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}T\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

(b) (5 points) Find the matrix representation of T.

$$\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}$$

(c) (5 points) What is the co-domain, range, and kernel of T? **Hint:** All of the calculations needed for this problem were done in part (a).

Co-domain $= \mathbb{R}^2$

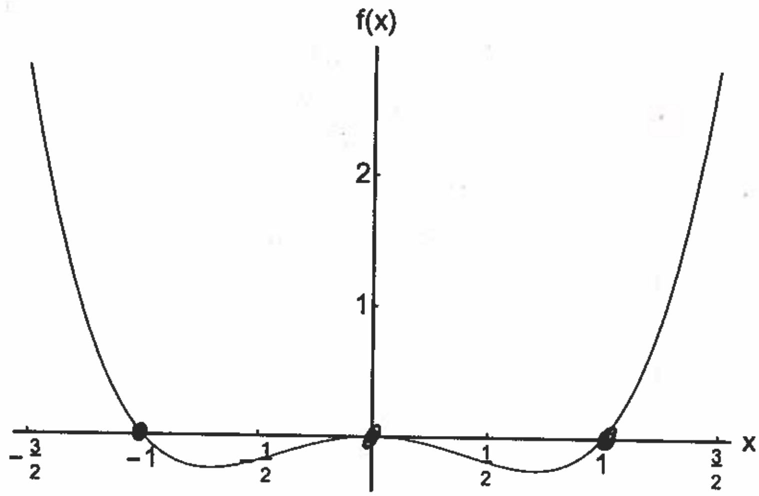
range $= \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

kernel $= \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

6. (10 points) Consider the following differential equation

$$\frac{dx}{dt} = f(x),$$

where $f(x)$ is plotted below.



(a) (5 points) **Short Answer:** On the figure, indicate the location of any fixed points, i.e. equilibrium points.

(b) (5 points) **Short Answer:** On one axis sketch the solutions curves for this differential equation as functions of time. Include enough solution curves so that they illustrate all possible qualitatively different possibilities.