Section 2.4: Linear Differential Equations.

$q_0(x)\, y' + q_1(x) y(x) = q_2(x)$

- Linear differential equation since first power in y and its derivatives.
- homogeneous if $q_2 = 0$
- inhomogeneous if $q_2 \neq 0$.

Equivalent form:

$$\boxed{\begin{array}{l} y' + p(x) y(x) = q(x). \\ y(x_0) = y_0 \end{array}}$$

How do we solve??

Example:

1. $e^x y' + e^x y = x^2, \quad y(0) = y_0$

$$\Rightarrow \frac{d}{dx}(e^x y) = x^2 \qquad \underline{\underline{\text{or}}} \qquad \int_{e^0 y_0}^{e^x y} d(e^x y) = \int_0^x x^2\,dx$$

$$\Rightarrow e^x y = \frac{x^3}{3} + c \qquad\qquad \Rightarrow y e^x - y_0 = \frac{x^3}{3}$$

$$\Rightarrow y = \left(\frac{x^3}{3} + c\right) e^{-x} \qquad\qquad y = \left(\frac{x^3}{3} + y_0\right) e^{x}.$$

$$y(0) = y_0 \Rightarrow y_0 = c$$

$$y(x) = \left(\frac{x^3}{3} + y_0\right) e^{-x}.$$

2. $y' = 2y + x$

$\Rightarrow y' - 2y = x$

We would like to use the product rule. Multiply both sides by $e^{f(x)}$.

$$e^{f(x)}y' - 2e^{f(x)}y = e^{f(x)}x$$

$$\Rightarrow \frac{d}{dx}\left(e^{f(x)}y\right) = e^{f(x)}x \qquad (*)$$

$$\Rightarrow e^{f(x)}y' + f'(x)e^{f(x)}y = e^{f(x)}x$$

$$f'(x) = -2$$

$$\Rightarrow f(x) = -2x$$

Therefore, substituting back into $(*)$ we have that:

$$\frac{d}{dx}\left(e^{-2x}y\right) = e^{-2x}x$$

$$\Rightarrow e^{-2x}y = \int xe^{-2x}dx$$

$$= -\frac{x}{2}e^{-2x} + \int \tfrac{1}{2}e^{-2x}dx$$

$$= -x/2\, e^{-2x} - 1/4\, e^{-2x} + C$$

$$\Rightarrow y = \frac{-x}{2} - \frac{1}{4} + Ce^{2x}.$$

3. $\frac{dy}{dx} - \frac{1}{x}y = 0$

$$e^{f(x)}\frac{dy}{dx} - \frac{1}{x}e^{f(x)}y = 0$$

$f'(x) = -1/x$

$f(x) = -\ln(x)$

Therefore, we obtain:

$$\frac{d}{dx}\left(e^{f(x)}y(x)\right)=0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y(x)\right)=0$$

$$\Rightarrow \frac{1}{x}y(x)=C$$

$$\Rightarrow y(x)=Cx.$$

This is also a separable equation.

$$\frac{dy}{dx}=\frac{1}{x}y$$

$$\Rightarrow \int \frac{1}{y}dy=\int \frac{1}{x}dx$$

$$\Rightarrow \ln(y)=\ln(x)+C$$

$$y=Cx.$$

4. $y'=xy-x$

$y(1)=2$

$$\Rightarrow \frac{dy}{dx}=x(y-1)$$

$$\Rightarrow \int_2^y \frac{1}{y-1}dy=\int_1^x x\,dx$$

$$\Rightarrow \ln(y-1)-\ln(1)=\frac{x^2}{2}-\frac{1}{2}$$

$$\Rightarrow y-1=\exp\left(\frac{x^2-1}{2}\right)$$

$$\Rightarrow y=1+\exp\left(\frac{x^2-1}{2}\right).$$

We also have that:

$$y' - xy = -x$$

$$e^{f(x)}y' - xe^{f(x)}y = -xe^{f(x)}$$

$$f'(x) = -x$$

$$f(x) = \frac{-x^2}{2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x^2/2}y\right) = -xe^{-x^2/2}$$

$$\int_{2e^{-1/2}}^{e^{-x^2/2}y} d\left(e^{-x^2/2}y\right) = \int_1^x -xe^{-x^2/2}\,dx \qquad u = x^2/2 \Rightarrow du = x\,dx$$

$$\Rightarrow e^{-x^2/2}y - 2e^{-1/2} = \int_{1/2}^{x^2/2} -e^{-u}\,du$$

$$\Rightarrow e^{-x^2/2}y - 2e^{-1/2} = e^{-x^2/2} - e^{-1/2}$$

$$\Rightarrow y = 1 + e^{-1/2}e^{x^2/2}$$

$$= 1 + \exp\left(\frac{x^2-1}{2}\right).$$