Section 4.1: Linear Differential Equations.

Second Order Linear Equations:

$$q_2(t)\frac{d^2y}{dt^2} + q_1(t)\frac{dy}{dt} + q_0(t)y = f(t)$$

$$y(t_0) = y_0$$

$$\left.\frac{dy}{dt}\right|_{t_0} = v_0$$

- Linear
- Second Order
- Homogeneous if $f = 0$
- Inhomogeneous if $f \neq 0$.

Theorem - Solutions to the homogeneous equation

$$q_2(t)\frac{d^2y}{dt^2} + q_1(t)\frac{dy}{dt} + q_0(t)y = 0 \quad (*)$$

$$y(t_0) = y_0$$

$$y'(t_0) = v_0$$

form a vector space of dimension 2.

proof:

1. We first prove subspace properties. Let y_1, y_2 be solutions to $(*)$ and let $y_3(t) = y_1 + y_2$. Then,

$$q_2(t)\frac{d^2y_3}{dt^2} + q_1(t)\frac{dy_3}{dt} + q_0(t)y_3$$

$$= q_2(t)\left(\frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}\right) + q_1(t)\left(\frac{dy_1}{dt} + \frac{dy_2}{dt}\right) + q_0(t)(y_1(t) + y_2(t))$$

$$= q_2\frac{d^2y_1}{dt^2} + q_1\frac{dy_1}{dt} + q_0 y_1 + q_2\frac{d^2y_2}{dt^2} + q_1\frac{dy_2}{dt} + q_0 y_2$$

$$= 0.$$

Also, if $y_4 = cy_1$ it follows that

$$q_2\frac{d^2y_4}{dt^2} + q_1\frac{dy_4}{dt} + q_0 y_4 = q_2 c\frac{d^2y_1}{dt^2} + q_1 c\frac{dy_1}{dt} + q_0 c y_1$$

$$= c\left(q_2\frac{d^2y_1}{dt^2} + q_1\frac{dy_1}{dt} + q_0 y_1\right)$$

$$= 0.$$

2. We now need to find a basis. Let $\bar{y}_1$ and $\bar{y}_2$ solve

$$q_2\frac{d^2\bar{y}_1}{dt^2} + q_1\frac{d\bar{y}_1}{dt} + q_0\bar{y}_1 = 0$$

$$\bar{y}_1(t_0) = 1$$

$$\bar{y}_1'(t_0) = 0$$

$$q_2\frac{d^2\bar{y}_2}{dt^2} + q_1\frac{d\bar{y}_2}{dt} + q_0\bar{y}_2 = 0$$

$$\bar{y}_2(t_0) = 0$$

$$\bar{y}_2'(t_0) = 1$$

We first show that $\bar{y}_1$ and $\bar{y}_2$ are linearly independent.

$$\Rightarrow c_1\bar{y}_1 + c_2\bar{y}_2 = 0$$

$$c_1\bar{y}_1' + c_2\bar{y}_2' = 0$$

Wronskian at t_0 is given by

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \neq 0.$$

We now show that $\bar{y}_1, \bar{y}_2$ span the solution space.

Suppose y solves

$$q_2 y'' + q_1 y' + q_0 y = 0$$
$$y(t_0) = y_0$$
$$y'(t_0) = V_0$$

Consider,

$$\bar{y} = y_0 \bar{y}_1 + V_0 \bar{y}_2$$
$$\bar{y}(t_0) = y_0 \bar{y}_1(t_0) + V_0 \bar{y}_2(t_0)$$
$$= y_0$$
$$\bar{y}'(t_0) = y_0 \bar{y}_1'(t_0) + V_0 \bar{y}_2'(t_0)$$
$$= V_0$$

Therefore,

$$y(t) = \bar{y} = y_0 \bar{y}_1(t) + V_0 \bar{y}_2(t)$$
$$\Rightarrow y \in \text{span}\{\bar{y}_1, \bar{y}_2\}.$$

Consequently, $\bar{y}_1, \bar{y}_2$ are a basis!

Example:

- Find a basis for the solution space for the equation:

$$y'' - y' - 2y = 0$$

- linear
- second order
- homogeneous
- constant coefficient

Guess: $y = e^{\lambda x}$

$$\Rightarrow \lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0$$
$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$

$\Rightarrow (\lambda-2)(\lambda+1)=0$

$\Rightarrow \lambda = 2, -1.$

Two solutions:

$e^{2x}, e^{-x}.$

Check linear independence:

$$c_1 e^{2x} + c_2 e^{-x} = 0$$
$$2c_1 e^{2x} - c_2 e^{-x} = 0$$

$$\det\left(\begin{bmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{bmatrix}\right) = -e^{x} - 2e^{x} = -3e^{x} \neq 0.$$

All solutions can be written the form

$$y(x) = c_1 e^{2x} + c_2 e^{-x}$$

- Solve

$$y'' - y' - 2y = 0$$
$$y(0) = 1$$
$$y'(0) = 0$$

$$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{-x}$$
$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x}$$

$$\Rightarrow y(0) = c_1 + c_2 = 1$$
$$y'(0) = 2c_1 - c_2 = 0$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 0 \end{array}\right] -2R1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -2 \end{array}\right]$$

$$\Rightarrow c_2 = 2/3$$
$$c_1 + 2/3 = 1$$
$$c_1 = 1/3$$

$$\Rightarrow y(x) = 1/3\, e^{2x} + 2/3\, e^{-x}.$$

Example:

$$-\frac{dy}{dt} + \frac{1}{2}y = 0$$

Guess:

$$y(t) = e^{\lambda t}$$

$$y'(t) = \lambda e^{\lambda t}$$

$$\Rightarrow \lambda e^{\lambda t} + \tfrac{1}{2}e^{\lambda t} = 0$$

$$\lambda = -\tfrac{1}{2}.$$

$$y(t) = e^{-\frac{1}{2}t}$$

All solutions are of the form

$$y(t) = ce^{-\frac{1}{2}t}.$$

$$-\frac{dy}{dt} + \frac{1}{2}y = 2\cos(t)$$

Guess:

$$y(t) = A\cos(t) + B\sin(t)$$

$$y'(t) = -A\sin(t) + B\cos(t)$$

$$-A\sin(t) + B\cos(t) + \tfrac{1}{2}A\cos(t) + \tfrac{1}{2}B\sin(t) = 2\cos(t)$$

$$\Rightarrow -A + \tfrac{1}{2}B = 0$$

$$\tfrac{1}{2}A + B = 2$$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 1 & 2 & 4 \end{array}\right] +2R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 5 & 8 \\ 1 & 2 & 4 \end{bmatrix}$$

$$B = \tfrac{8}{5}$$

$$A + \tfrac{16}{5} = 4$$

$$\Rightarrow A = \tfrac{4}{5}$$

$$\Rightarrow y(t) = \tfrac{4}{5}\cos(t) + \tfrac{8}{5}\sin(t).$$

- Solve:

$$\frac{dy}{dt} + \frac{1}{2} y = 2\cos(t)$$

$$y(0) = -1.$$

Generic solution =

$$y = \underbrace{y_H}_{\text{Homogeneous solution}} + \underbrace{y_p}_{\text{particular solution.}}$$

$$y(t) = \underbrace{ce^{-\frac{1}{2}t}}_{y_H} + \underbrace{\tfrac{4}{5}\cos(t) + \tfrac{8}{5}\sin(t)}_{y_p}$$

$$y(0) = c + \tfrac{4}{5} = -1$$

$$\Rightarrow c = -\tfrac{9}{5}.$$

$$\Rightarrow y(t) = -\tfrac{9}{5} e^{-\frac{1}{2}t} + \tfrac{4}{5}\cos(t) + \tfrac{8}{5}\sin(t).$$