Section 5.3: Matrices for Linear Transformations

$V = \mathbb{R}^n \to$ dimension n
$V = P_n \to$ dimension $n+1$
} finite dimensional

Linear transformations between finite dimensional vector spaces can always be represented by matrices.

Example:

1. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$T(\vec{e}_2) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

$$T(\vec{e}_3) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Therefore,

Input basis

$$\begin{aligned} T(v) &= T(x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3) \\ &= xT(\vec{e}_1) + yT(\vec{e}_2) + zT(\vec{e}_3) \\ &= x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Output basis.

$$\begin{aligned} &= (x+2y+3z)\vec{e}_1 + (2x+3y+1z)\vec{e}_2 + (3x+y+2z)\vec{e}_3 \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

2. $T: P_2 \to P_2$ defined by

$T(ax^2+bx+c) = 2ax+b.$

Standard basis: $\{x^2, x, 1\}$.

$T(x^2) = 2x, \quad T(x) = 1, \quad T(1) = 0$

$$T(\vec{v}) = T(ax^2+bx+c)$$
$$= aT(x^2)+bT(x)+T(c)$$
$$= 2ax+b$$
$$= 2a\vec{e}_2 + b\vec{e}_3$$

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Check! $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{Matrix Representation}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}}_{\text{Matrix Representation}}$

Example:

$T(x, y, z) = (5x+y, 3x+2y)$

$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Find

$[T]_\alpha^\alpha$

$[T]_\beta^\beta$

$[T]_\alpha^\beta$

$[T]_\beta^\alpha$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_\alpha\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_\alpha$

$\Rightarrow \boxed{[T]_\alpha^\alpha = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_\alpha\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha$

Check: $[T]_\alpha^\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x+y \\ 3x+2y \end{bmatrix}$

What about $[T]_\beta^\beta$?

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}_\alpha\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}_\beta\right) = \begin{bmatrix}6\\5\end{bmatrix}_\alpha$$

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}_\alpha\right) = T\left(\begin{bmatrix}0\\1\end{bmatrix}_\beta\right) = \begin{bmatrix}4\\1\end{bmatrix}_\alpha$$

$$\Rightarrow c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}6\\5\end{bmatrix},\quad c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|cc}1 & 1 & 6 & 4\\ 1 & -1 & 5 & 1\end{array}\right] \begin{array}{l}\\ -R1\end{array} \Rightarrow \left[\begin{array}{cc|cc}1 & 1 & 6 & 4\\ 0 & -2 & -1 & -3\end{array}\right] \begin{array}{l}\\ /-2\end{array}$$

$$\Rightarrow \left[\begin{array}{cc|cc}1 & 1 & 6 & 4\\ 0 & 1 & 1/2 & 3/2\end{array}\right] \begin{array}{l}-R2\\ \end{array} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 11/2 & 5/2\\ 0 & 1 & 1/2 & 3/2\end{array}\right]$$

Therefore,

$$\begin{bmatrix}6\\5\end{bmatrix}_\alpha = \begin{bmatrix}11/2\\1/2\end{bmatrix}_\beta,\quad \begin{bmatrix}4\\1\end{bmatrix}_\alpha = \begin{bmatrix}5/2\\3/2\end{bmatrix}$$

Consequently,

$$\boxed{[T]_\beta^\beta = \begin{bmatrix}11/2 & 5/2\\ 1/2 & 3/2\end{bmatrix}.}$$

Check:

$$[T]_\beta^\beta\begin{bmatrix}1\\0\end{bmatrix}_\beta = \begin{bmatrix}11/2\\1/2\end{bmatrix}_\beta = 11/2\begin{bmatrix}1\\1\end{bmatrix}_\alpha + 1/2\begin{bmatrix}1\\-1\end{bmatrix}_\alpha = \begin{bmatrix}6\\5\end{bmatrix}_\alpha.$$

What about $[T]_\alpha^\beta$?

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}_\alpha\right)=\begin{bmatrix}5\\3\end{bmatrix}_\alpha,\quad T\left(\begin{bmatrix}0\\1\end{bmatrix}_\alpha\right)=\begin{bmatrix}1\\2\end{bmatrix}_\alpha$$

$$c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix},\quad c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow\left[\begin{array}{cc:cc}1&1&5&1\\1&-1&3&2\end{array}\right]\begin{matrix}\\-R1\end{matrix}\Rightarrow\left[\begin{array}{cc:cc}1&1&5&1\\0&-2&-2&1\end{array}\right]\begin{matrix}\\/-2\end{matrix}\Rightarrow\left[\begin{array}{cc:cc}1&1&5&1\\0&1&1&-\tfrac{1}{2}\end{array}\right]\begin{matrix}-R2\\ \\ \end{matrix}$$

$$\Rightarrow\left[\begin{array}{cc:cc}1&0&4&\tfrac{1}{2}\\0&1&1&-\tfrac{1}{2}\end{array}\right]$$

$$\Rightarrow\begin{bmatrix}5\\3\end{bmatrix}_\alpha=\begin{bmatrix}4\\1\end{bmatrix}_\beta,\quad\begin{bmatrix}1\\2\end{bmatrix}_\alpha=\begin{bmatrix}\tfrac{1}{2}\\-\tfrac{1}{2}\end{bmatrix}_\beta$$

$$\Rightarrow\boxed{[T]_\alpha^\beta=\begin{bmatrix}4&\tfrac{1}{2}\\1&-\tfrac{1}{2}\end{bmatrix}}$$

Check:

$$[T]_\alpha^\beta\begin{bmatrix}1\\0\end{bmatrix}_\alpha=\begin{bmatrix}4\\1\end{bmatrix}_\beta=4\begin{bmatrix}1\\1\end{bmatrix}_\alpha+\begin{bmatrix}1\\-1\end{bmatrix}_\alpha=\begin{bmatrix}5\\3\end{bmatrix}_\alpha$$

What about $[T]_\beta^\alpha$?

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}_\alpha\right)=T\left(\begin{bmatrix}1\\0\end{bmatrix}_\beta\right)=\begin{bmatrix}6\\5\end{bmatrix}_\alpha,\quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}_\alpha\right)=T\left(\begin{bmatrix}0\\1\end{bmatrix}_\beta\right)=\begin{bmatrix}4\\1\end{bmatrix}_\alpha$$

$$\Rightarrow\boxed{[T]_\beta^\alpha=\begin{bmatrix}6&4\\5&1\end{bmatrix}}$$

Change of Basis

$$\alpha = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}, \quad \beta = \left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

Convert:

$\vec{V}_\alpha = \begin{bmatrix}1\\1\\1\end{bmatrix}_\alpha$ into β representation, $\vec{V}_\beta$

$$\vec{V}_\beta = [I]_\alpha^\beta \vec{V}_\alpha \quad \text{and} \quad [I]_\beta^\alpha \vec{V}_\beta = \vec{V}_\alpha$$

$$\Rightarrow [I]_\beta^\alpha = \begin{bmatrix}1&1&1\\1&-1&1\\2&1&1\end{bmatrix} \Rightarrow [I]_\alpha^\beta = \left([I]_\beta^\alpha\right)^{-1}$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&-1&1&0&1&0\\2&1&1&0&0&1\end{array}\right]\begin{array}{l}\\-R1\\-2R1\end{array} \Rightarrow \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-2&0&-1&1&0\\0&-1&-1&-2&0&1\end{array}\right]\begin{array}{l}\\/-2\\/-1\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&1&1&2&0&-1\end{array}\right]\begin{array}{l}-R2\\\\-R2\end{array} \Rightarrow \left[\begin{array}{ccc|ccc}1&0&1&\frac{1}{2}&0&0\\0&1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&0&1&\frac{3}{2}&\frac{1}{2}&-1\end{array}\right]\begin{array}{l}-R3\\\\\\\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1&0&0&-1&0&0\\0&1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&0&1&\frac{3}{2}&\frac{1}{2}&-1\end{array}\right]$$

Columns are basis vectors.

Therefore,

$$[I]_\alpha^\beta = P = \begin{bmatrix}-1&0&1\\\frac{1}{2}&-\frac{1}{2}&0\\\frac{3}{2}&\frac{1}{2}&-1\end{bmatrix}, \quad [I]_\beta^\alpha = \begin{bmatrix}1&1&1\\1&-1&1\\2&1&1\end{bmatrix}$$

$$[I]_\alpha^\beta \begin{bmatrix}1\\1\\1\end{bmatrix}_\alpha = \begin{bmatrix}0\\0\\1\end{bmatrix}_\beta$$

Check:

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_\beta = \begin{bmatrix}1\\1\\1\end{bmatrix}_\alpha$$

Change of Basis for matrix:

Convert $[T]_\alpha^\alpha$ into $[T]_\beta^\beta$

$$[T]_\beta^\beta \vec{v}_\beta = \vec{w}_\beta$$

$$\Rightarrow [T]_\beta^\beta [I]_\alpha^\beta \vec{v}_\alpha = [I]_\alpha^\beta \vec{w}_\alpha$$

$$\Rightarrow \left([I]_\alpha^\beta\right)^{-1} [T]_\beta^\beta [I]_\alpha^\beta \vec{v}_\alpha = \vec{w}_\alpha$$

$$\Rightarrow [T]_\alpha^\alpha = \left([I]_\alpha^\beta\right)^{-1} [T]_\beta^\beta [I]_\alpha^\beta$$

$$[T]_\beta^\beta = \left([I]_\alpha^\beta\right) [T]_\alpha^\alpha \left([I]_\alpha^\beta\right)^{-1}$$

$$[T]_\beta^\beta = \left([I]_\beta^\alpha\right)^{-1} [T]_\alpha^\alpha [I]_\beta^\alpha$$

$$\boxed{[T]_\beta^\beta = P^{-1} [T]_\alpha^\alpha P}$$

Recall:

P is the matrix with columns given by the basis β.