

Lecture 2.0: Eigenvalues and Eigenvectors:

Diagonal Matrices:

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = A = \left[A\vec{e}_1 \mid \dots \mid A\vec{e}_n \right] = \left[d_1\vec{e}_1 \mid \dots \mid d_n\vec{e}_n \right]$$

$A\vec{e}_i = d_i\vec{e}_i \rightarrow$ Similar to matrix multiplication.

Eigenvalues and Eigenvectors

- A - $n \times n$ matrix

- \vec{v} is an eigenvector with corresponding eigenvalue if

$$A\vec{v} = \lambda\vec{v}$$

Example:

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

If \vec{v} is an eigenvector with eigenvalue λ then.

$$A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0.$$

This has a nontrivial solution if:

$$\det \begin{pmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$\Rightarrow \lambda_1 = 4$ and $\lambda_2 = -1$ are eigenvalues.

How do we find eigenvectors?

1. For λ_1 :

$$A - \lambda_1 I = \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}$$

$$\Rightarrow (A - \lambda_1 I) \vec{v}_1 = 0$$

$$\Rightarrow \vec{v}_1 \in \text{NS}(A - \lambda_1 I) = E_{\lambda_1}$$

eigenspace of $\lambda_1 = 4$

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = -y$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} -y \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2. For λ_2 :

$$A - \lambda_2 I = \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \xrightarrow{+R_1} \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = \frac{3}{2}y$$

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{2}y \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$ with associated eigenvectors

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The eigenspaces are

$$E_{-4} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, E_4 = \text{span} \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

For this problem define

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \beta = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

$$[I]_{\beta}^{\alpha} = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ -1 & 3 & 1 & 0 \end{array} \right] + R1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 5 & 1 & 1 \end{array} \right] \frac{1}{5} \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right] -2R2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right]$$

$$[I]_{\alpha}^{\beta} = ([I]_{\beta}^{\alpha})^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$[T]_{\alpha}^{\alpha} \vec{v}_{\alpha} = \vec{w}_{\alpha} \rightarrow \text{Normal Matrix multiplication.}$$

$$[T]_{\alpha}^{\alpha} [I]_{\beta}^{\alpha} \vec{v}_{\beta} = [I]_{\beta}^{\alpha} \vec{w}_{\beta}$$

$$\Rightarrow ([I]_{\beta}^{\alpha})^{-1} [T]_{\alpha}^{\alpha} [I]_{\beta}^{\alpha} \vec{v}_{\beta} = \vec{w}_{\beta} = [T]_{\beta}^{\beta} \vec{v}_{\beta}$$

$$\Rightarrow \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1. Eigenvalues:

$$\lambda_1 = 2, \lambda_2 = -1 \quad A\vec{v}_\lambda = \lambda\vec{v}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(\lambda+1)^2$$

2. Eigenspaces:

a.) $\lambda_1 = 2$

$$(A - \lambda_1 I) = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Rightarrow \text{NS}(A - \lambda_1 I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

b.) $\lambda_2 = -1$

$$(A - \lambda_2 I) = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{v} \in \text{NS}(A - \lambda_2 I)$ then

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y/3 - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspaces:

$$E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_{\lambda_2} = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example:

Determine the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$1. \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix}$$

$$= 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$= \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$$

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$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i = \bar{\lambda}_1 \rightarrow \text{complex conjugate.}$$

$$2. A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{-i} \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \xrightarrow{-R1} \\ = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$NS(A - \lambda_1 I) = \text{span} \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$$

$$NS(A - \lambda_2 I) = \text{span} \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$$