Lecture 22: Homogeneous Systems

Example:

$$\dot{x} = -3x \qquad \Rightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}\vec{x}, \quad \vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$\dot{y} = 4y$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \rightarrow \text{Linear System}$$

If $\vec{x}_1, \vec{x}_2$ are solutions than so is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}_1}{dt} + c_2\frac{d\vec{x}_2}{dt} = c_1A\vec{x}_1 + c_2A\vec{x}_2 = A(c_1\vec{x}_1 + c_2\vec{x}_2)$$
$$= A\vec{x}.$$

$$\frac{dx}{dt} = -3x \Rightarrow x(t) = c_1e^{-3t}$$

$$\frac{dy}{dt} = 4y \Rightarrow y(t) = c_2e^{4t}$$

$$\Rightarrow \vec{x} = c_1\begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{4t} \end{bmatrix}.$$

Example:

$$\frac{dx}{dt} = A\vec{x}$$

If λ_1 is an eigenvalue with corresponding eigenvector $\vec{v}_1$ it follows that:

$$\vec{x}_1 = e^{\lambda t}\vec{v}_1$$

is a solution.

$$\frac{d\vec{x}_1}{dt} = \lambda e^{\lambda t}\vec{v}_1 = e^{\lambda t}\lambda\vec{v}_1 = e^{\lambda t}A\vec{v}_1 = A(e^{\lambda t}\vec{v}_1) = A\vec{x}_1.$$

If there are n linearly ind. eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n.$$

__Example:__

$\dot{\vec{x}} = A\vec{x}$ where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are

$\lambda_1 = -1$, $\lambda_2 = -2$

__If $\lambda_1 = -1$:__

$$A - \lambda_1 I = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

__If $\lambda_2 = -2$__

$$A - \lambda_2 I = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The general solution is therefore,

$$y = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-2t} \\ c_2 e^{-2t} \end{bmatrix}$$

__Example:__

$\dot{\vec{x}} = A\vec{x}$.

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

__Eigenvalues__ of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = 1 - 2\lambda + \lambda^2 + 1$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Eigenvectors of A:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \cdot i = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} - R1 = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow y = \text{anything}$

$x = iy$

eigenvectors $= \begin{bmatrix} i \\ 1 \end{bmatrix}$

General Solution is

$$\vec{x}(t) = c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} i & -i & 1 \\ 1 & 1 & 0 \end{array}\right] \times i$$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & i \\ 1 & 1 & 0 \end{array}\right] + R1 = \left[\begin{array}{cc|c} -1 & 1 & i \\ 0 & 2 & i \end{array}\right]$$

$\Rightarrow c_2 = i/2$

$-c_1 + i/2 = i$

$c_1 = -i/2$

$$\Rightarrow \vec{x}(t) = e^{(1+i)t} \begin{bmatrix} 1/2 \\ -i/2 \end{bmatrix} + e^{(1-i)t} \begin{bmatrix} 1/2 \\ i/2 \end{bmatrix}$$

$$= \frac{1e^t}{2} \left[(\cos(t) + i\sin(t)) \begin{bmatrix} -1 \\ -i \end{bmatrix} + (\cos(t) - i\sin(t)) \begin{bmatrix} 1 \\ i \end{bmatrix} \right]$$

$$= \frac{e^t}{2} \begin{bmatrix} +\cos(t) + i\sin(t) + \cos(t) - i\sin(t) \\ -i\cos(t) + \sin(t) + i\cos(t) + \sin(t) \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Example:

$$\dot{\vec{x}} = A\vec{x}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

One solution is $e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} \frac{dx}{dt} &= x+y \\ \frac{dy}{dt} &= y \end{aligned}\right\rangle \frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{dy}{dt} = \frac{dx}{dt} + y = \frac{dx}{dt} + \frac{dx}{dt} - x$$

$$\Rightarrow \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0$$

Repeated Eigenvectors application