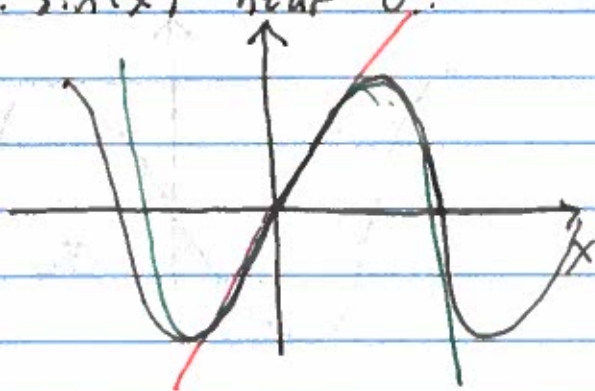


## Lecture 23: Power Series Solutions.

How do you approximate a function by a polynomial?

1.  $\sin(x)$  near 0:



Approximate by a tangent line.

a.)  $y = mx + b$

$$m = f'(0) = \cos(0) = 1$$

$$b = y\text{-int} = 0.$$

$\Rightarrow y_1 = x \sim$  best linear approximation.

b.) What about best quadratic approximation?

$$y = c_0 + c_1x + c_2x^2$$

$$y(0) = f(0) \Rightarrow c_0 = f(0) = 0$$

$$y'(0) = f'(0) \Rightarrow c_1 = f'(0) = 1$$

$$y''(0) = f''(0) \Rightarrow 2c_2 = f''(0) = 0$$

$\Rightarrow y_2 = x \sim$  best quadratic approximation.

c.) What about best cubic approximation?

$$y = c_0 + c_1x + c_2x^2 + c_3x^3$$

$$y'''(0) = f'''(0) \Rightarrow 6c_3 = f'''(0) = -1$$

$\Rightarrow y_3 = x - \frac{1}{6}x^3 \sim$  best cubic approximation.

2.  $\sin(x)$  near  $x = \pi/2$

$$\sin(x) = c_0 + c_1(x - \pi/2) + c_2(x - \pi/2)^2 + c_3(x - \pi/2)^3 + \dots$$

$$- \sin(\pi/2) = 1 \Rightarrow c_0 = 1$$

$$- \sin'(x) = \cos(x)$$

$$\Rightarrow \cos(\pi/2) = 0 = c_1$$

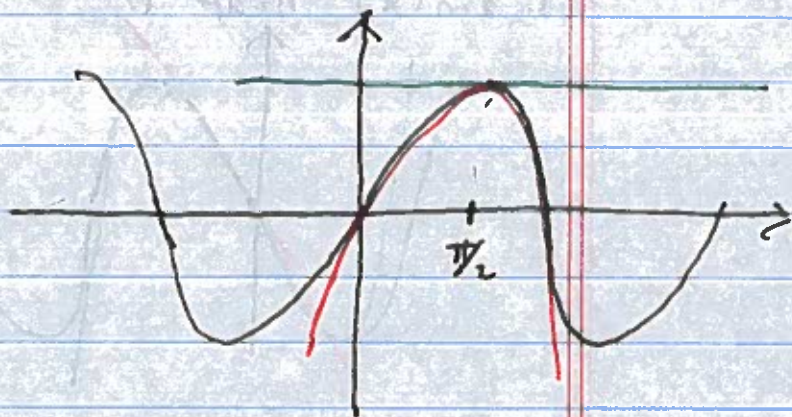
$$- \sin''(x) = -\sin(x)$$

$$\Rightarrow -\sin(\pi/2) = -1 = 2c_2$$

$$\Rightarrow c_2 = -1/2$$

$$\sin(x) \approx 1 - \frac{1}{2}(x - \pi/2)^2$$

best quadratic approximation



Power Series Approximation of a function

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

center of approximation.

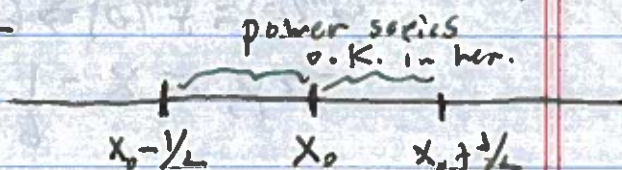
Does this function make sense??

Ratio Test!

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x - x_0| < 1$$

Let  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ . Therefore,

$$|x - x_0| < \frac{1}{L} \Leftrightarrow \frac{1}{L} = \text{radius of convergence.}$$
$$\Rightarrow x_0 - \frac{1}{L} < x < x_0 + \frac{1}{L}$$



Example:

$$- f(x) = \frac{1}{1+x} = (1+x)^{-1}, f'(x) = -(1+x)^{-2}, f''(x) = 2(1+x)^{-3}, \dots$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

$$f''(x) = 2c_2 + 6c_3 x + \dots$$

↓

$$f(0) = 1 = c_0$$

$$f'(0) = -1 = c_1$$

$$f''(0) = 2 = 2c_2 \Rightarrow c_2 = 1$$

$$f(x) = 1 - x + x^2 - x^3 + \dots$$

Radius of convergence?

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 = L$$

$$\Rightarrow \rho = 1$$

$$\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad \text{if } -1 < x < 1$$

$$- f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \quad \text{if } -1 < x < 1$$

Important Taylor Series:

$$- \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots, \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$- \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots, \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$- e^x = 1 + x + \frac{x^2}{2!} + \dots, \quad \ln(1+x) = -1 + 2x - 3x^2 + \dots$$

Example:

Solve:

$$y' = (x-1)^2 y \approx (x-1)^2 y$$

$$y(1) = -1$$

Guess

$$y(x) = c_0 + c_1(x+1) + c_2(x+1)^2 + c_3(x+1)^3 + \dots$$

$$y'(x) = c_1 + 2c_2(x+1) + 3c_3(x+1)^2 + \dots$$

$$\Rightarrow (c_1 + 2c_2(x+1) + 3c_3(x+1)^2 + \dots) = (x-1)^2 (c_0 + c_1(x+1) + c_2(x+1)^2 + c_3(x+1)^3 + \dots)$$

Gather powers:

$$-(x+1)^0: c_1 = 0$$

$$-(x+1)^1: 2c_2 = 0$$

$$-(x+1)^2: 3c_3 = c_0$$

$$-(x+1)^3: 4c_4 = c_1$$

$$-(x+1)^4: 5c_5 = c_2$$

Also:

$$y(1) = -1$$

$$\Rightarrow c_0 = -1$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow c_3 = -\frac{1}{3}$$

$$\Rightarrow c_4 = 0$$

$$\Rightarrow c_5 = 0$$

$$\Rightarrow c_6 = -\frac{1}{3} \cdot 6 = -\frac{1}{2} \cdot \frac{1}{2}$$

$$c_9 = -\frac{1}{3} \cdot \frac{1}{1 \cdot 2 \cdot 3}$$

$$c_{12} = -\frac{1}{3} \cdot \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$\Rightarrow y(x) = -1 - \frac{1}{3}x^3 - \frac{1}{3} \cdot \frac{1}{2!}x^6 - \frac{1}{3} \cdot \frac{1}{3!}x^9 - \frac{1}{3} \cdot \frac{1}{3!}x^{12} + \dots$$

$$= -1 - \frac{1}{3}(x^3 + \frac{1}{2!}x^6 + \frac{1}{3!} + \dots)$$

$$= -1 - \frac{1}{3}(e^{x^3} - 1)$$

$$= -\frac{2}{3} - \frac{1}{3}e^{x^3}$$