Section 1.6: Properties of Determinants

1. A is invertible $\Leftrightarrow \det(A) \neq 0$.
2. $\det(A \cdot B) = \det(A)\det(B)$.
3. $\det(A^{-1}) = \dfrac{1}{\det(A)}$

~~proof~~:

$$\det(A \cdot A^{-1}) = \det(I)$$
$$\Rightarrow \det(A)\det(A^{-1}) = \det(I)$$
$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

4. $\det(A^T) = \det(A)$

~~proof~~:

Expanding along row = expanding along column.

Example:

For what values of λ does the following system have non-trivial solutions

$$x + 3y = \lambda x$$
$$4x + 2y = \lambda y$$

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}}_{A - \lambda I}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda - 5)(\lambda + 2)$$

Therefore, if $\lambda = 5, \lambda = -2$ this system has nontrivial solutions.

5. $\det(cA) = c^n \det(A)$.

proof

Multiply every row by c.

Cramer's Rule:

Solve:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & b/a & e/a \\ 1 & d/c & f/c \end{array}\right] -R1 \Rightarrow \left[\begin{array}{cc|c} 1 & b/a & e/a \\ 0 & \frac{da-bc}{ac} & \frac{fa-ec}{ac} \end{array}\right] ac/da-bc$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & b/a & e/a \\ 0 & 1 & fa-ec/da-bc \end{array}\right]$$

$$y = \frac{fa-ec}{da-bc} = \det\left(\begin{bmatrix} a & e \\ c & f \end{bmatrix}\right) / \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Also,

$$x = \det\left(\begin{bmatrix} e & b \\ f & d \end{bmatrix}\right) / \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$