Lecture 7: Vector Spaces

Definition - A set V is a vector space if there are operations of addition and scalar multiplication on V such that the following hold.

For all $\vec{u}, \vec{v}, \vec{w} \in V$, $a, b \in \mathbb{R}$:

1. $\vec{u} + \vec{v} \in V$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
4. There exists $0 \in V$, so that $0 + \vec{u} = \vec{u}$.
5. There is $(-\vec{u}) \in V$, such that $\vec{u} + (-\vec{u}) = \vec{0}$.
6. $a\vec{u} \in V$.
7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
8. $(a+b)\vec{u} = a\vec{u} + b\vec{v}$
9. $a(b\vec{u}) = (ab)\vec{u}$.
10. $1\vec{u} = \vec{u}$.

Examples

1. vectors in $\mathbb{R}^n$:

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

is a vector space.

2. 2x2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a vector space.

3. Polynomials of degree n is a vector space.

4. Functions are a vector space.

5. $\mathbb{R}^+$ with $x \oplus y = xy$, $c \odot x = x^c$ is a vector space.

<u>Example:</u>

1. $(x_1, x_2) + (y_1, y_2) = (x_1 + x_2 + 1, y_1 + y_2)$ (Not a vector space)

2. $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$

   $a\vec{u} = (au_1, 0)$ (Not a vector space).

3. $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ (Not a vector space).