

Section 2.3: Linear Independence and Bases

Definition - The vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if one of the vectors can be written as a linear combination of the others i.e.,

$$\vec{v}_1 = c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_n \vec{v}_n \rightarrow \text{Useful for understanding}$$

i.e. \vec{v}_1 depends on the others.

Equivalently, $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if there are scalars c_1, \dots, c_n not all zero so that

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}, \rightarrow \text{Useful for problems.}$$

If $\vec{v}_1, \dots, \vec{v}_n$ are not linearly dependent, they are linearly independent.

Examples

1. $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix}$, are linearly dependent.

$$3\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \begin{bmatrix} 6 \\ -3 \\ 0 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{v}_3 = 3\vec{v}_1 + \vec{v}_2 \quad (\vec{v}_3 \text{ depends on } \vec{v}_1 \text{ and } \vec{v}_2).$$

* purple depends on red and blue.

2. $p_1 = 1-x$, $p_2 = 5+3x-2x^2$, $p_3 = 1+3x-x^2$

are linearly dependent

$$3p_1 - p_2 + 2p_3 = 0$$

$$\Rightarrow p_2 = 3p_1 + 2p_3$$

3. The vectors

$$\vec{i} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

4. Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

linearly dependent or linearly independent.

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 + 5c_2 + 3c_3 = 0$$

$$-2c_1 + 6c_2 + 2c_3 = 0$$

$$3c_1 - c_2 + c_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow{\substack{+2R_1 \\ -3R_1}} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

There are nontrivial solutions

\Rightarrow Linearly dependent.

$$\det(A) = 1 \cdot (6+2) - 5(-2-6) + 3(2-18)$$

$$= 8 + 40 - 48$$

$$= 0.$$

Let's try to solve for one of the vectors

c_3 arbitrary

$$16c_2 + 8c_3 = 0$$

$$c_2 = -\frac{1}{2}c_3$$

finally

$$c_1 + 5c_2 + 3c_3 = 0$$

$$c_1 - \frac{5}{2}c_3 + 3c_3 = 0$$

$$c_1 = -\frac{1}{2}c_3$$

$$\Rightarrow -\frac{1}{2}c_3 \vec{v}_1 - \frac{1}{2}c_3 \vec{v}_2 + c_3 \vec{v}_3 = 0$$

$$\Rightarrow \vec{v}_3 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$$

Definition - We say that the vectors $\vec{v}_1, \dots, \vec{v}_n$ are a basis for a vector space V if both of the following are satisfied:

1. $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.
2. $\vec{v}_1, \dots, \vec{v}_n$ span V .

Example:

1. Show that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

forms a basis for \mathbb{R}^3 .

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{/-2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes, linearly indy, also a basis.

2. $p_1 = x^2 + x - 3$, $p_2 = x - 5$, $p_3 = 3$

$$c_1(x^2 + x - 3) + c_2(x - 5) + 3c_3 = 0$$

$$c_1 = 0$$

$$c_1 + c_2 = 0$$

$$-3c_1 - 5c_2 + 3c_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & -5 & 3 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 3 \end{bmatrix}$$

$$\Rightarrow c_1 = 0$$

$c_2 = 0 \Rightarrow$ linear independence, basis.

$$c_3 = 0$$

$$3. p_1 = x^2 + x - 1, p_2 = x^2 - x + 1$$

Suppose

$$ax^2 + bx + c = c_1 p_1 + c_2 p_2$$

$$ax^2 + bx + c = (c_1 + c_2)x^2 + (c_1 - c_2)x + (c_2 - c_1)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 1 & b \\ -1 & 1 & 1 & c \end{array} \right] \begin{array}{l} \\ -R1 \\ +R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 0 & a-b \\ 0 & 2 & 0 & a+c \end{array} \right]$$

$$\Rightarrow c_2 = \frac{b-a}{2}, c_2 = \frac{a+c}{2}$$

Not consistent \Rightarrow Not a basis.

$$4. \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$c_1 - c_2 + 2c_3 = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right] -R1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{array} \right]$$

$c_3 = 2c_2 \Rightarrow$ linearly dependent \Rightarrow not a basis.