

Section 2.4: Dimension, nullspace, row space, column space

Lemma:

If $\vec{v}_1, \dots, \vec{v}_n$ are a basis and $\vec{w}_1, \dots, \vec{w}_m \in V$ with $m > n$ then $\vec{w}_1, \dots, \vec{w}_m$ are linearly dependent. If $m < n$ then $\vec{w}_1, \dots, \vec{w}_m$ do not span V .

Definition - If a vector space V has a basis of n vectors, we say the dimension of V is n .

Example:

1. $\dim(\mathbb{R}^n) = n \Rightarrow \text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

2. $\dim(\mathbb{R}^{n \times m}) = nm \Rightarrow \text{basis} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

3. $\dim(P_n) = n+1 \Rightarrow \text{basis} = \{1, x, x^2, \dots, x^n\}$

Theorem - Suppose V is a vector space of dimension n .

1. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ forms a basis for V .

2. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ span V , then $\{\vec{v}_1, \dots, \vec{v}_n\}$ form a basis for V .

Example:

$\{x^2-1, x^2+1, x+1\}$ is a basis for P_2 .

linear ind.:

$$c_1(x^2-1) + c_2(x^2+1) + c_3(x+1) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{+R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 \text{ only solution}$$

Nullspace

$$NS(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = 0 \}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 2 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_5 = \text{anything}$

$$x_2 - x_3 - x_4 - x_5 = 0$$

$$\Rightarrow x_2 = x_3 + x_4 + x_5$$

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$

$$x_1 - 2(x_3 + x_4 + x_5) - x_3 + 3x_4 = 0$$

$$\Rightarrow x_1 = 3x_3 - x_4 + 2x_5$$

If $\vec{v} \in NS(A)$ then

$$\vec{v} = \begin{bmatrix} 3x_3 - x_4 + 2x_5 \\ x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Rightarrow x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow NS(A) = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim(NS(A)) = 3.$$

Coordinates:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = 1\vec{e}_1 + 3\vec{e}_2 + 7\vec{e}_3$$

Express \vec{v} relative to the basis β :

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 - c_3 = 1 \\ c_2 + c_3 = 3 \\ c_1 + c_2 + c_3 = 7 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 7 \end{array} \right] -R1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{array} \right]$$

$$\Rightarrow \begin{aligned} c_3 &= 3 \\ c_2 + 3 &= 3 \\ c_2 &= 0 \\ c_1 + 0 - 3 &= 1 \\ c_1 &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}_\beta$$

Color basis.

One basis

- red, blue, yellow

Second Basis

- red, blue, green

Third Basis

- cyan, magenta, yellow

orange = red + yellow
= red + green + blue mix.

Example:

$$\vec{v} = x + 1$$

$$\gamma = \{x^2 + x - 3, x - 5, 3\}$$

$$x + 1 = c_1(x^2 + x - 3) + c_2(x - 5) + 3c_3$$

$$\Rightarrow 0 = c_1 x^2$$

$$x = c_1 x + c_2 x$$

$$1 = -3c_1 - 5c_2 + 3c_3$$

$$\Rightarrow c_1 = 0$$

$$c_2 = 1$$

$$1 = -5 + 3c_3$$

$$\Rightarrow c_3 = 3$$

$$\Rightarrow x + 1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \gamma$$

Row space:

$RS(A) =$ space spanned by rows of A

$$RS\left(\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\right\}$$

$$\dim(RS(A)) = 2.$$

$$\text{Rank}(A) = \dim(RS(A))$$

$$\text{Nullity}(A) = \dim(NS(A))$$

Theorem - $\text{Rank}(A) + \text{Nullity}(A) = \#$ of columns of A .

Column Space

$CS(A) =$ space spanned by columns of A

$$CS\left(\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix}\right) = RS\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 4 & 1 \\ 0 & 1 & -2 \end{bmatrix}\right) \begin{array}{l} -2R_1 \\ +R_1 \\ -3R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{array}{l} \\ \\ +R_2 \\ +R_2 \\ +R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow CS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right\}$$

Range(A)

$$\text{Ra}(A) = \{ \vec{w} : \text{there exists } \vec{x} \text{ such that } A\vec{x} = \vec{w} \}$$

$$\text{Ra}(A) = \text{CS}(A)$$

proof:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ c_1 & \dots & c_n \\ | & & | \end{bmatrix} = A \Rightarrow A\vec{e}_i = \vec{c}_i$$

If $\vec{w} \in \text{Ra}(A) \Rightarrow$ there exists \vec{v} such that

$$A\vec{v} = \vec{w}$$

$$\Rightarrow A(a_1\vec{e}_1 + \dots + a_n\vec{e}_n) = \vec{w}$$

$$\Rightarrow a_1 A\vec{e}_1 + \dots + a_n A\vec{e}_n = \vec{w}$$

$$\Rightarrow a_1 \vec{c}_1 + \dots + a_n \vec{c}_n = \vec{w}$$

$$\Rightarrow \vec{w} \in \text{CS}(A)$$