MST 750
Spring 2022
Exam #2
03/30/22

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	20	
2	10	
3	15	
4	20	
5	15	
6	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Consider the one parameter family of mappings $\varphi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\varphi_t(x) = (x-1)e^{-t} + 1.$$

(a) (5 points) **Short Answer: Briefly** write down the two properties that φ_t must satisfy in order to be a complete flow.

1. $\varphi_0(x) = x$
2. $\varphi_{s+t}(x) = \varphi_s \circ \varphi_t(x)$

(b) (10 points) Show that φ_t is a complete flow by showing that it satisfies the above two properties.

1. $\varphi_0(x) = (x-1)e^{-0} + 1 = x.$
2. $\varphi_{s+t}(x) = (x-1)e^{-(t+s)} + 1$

$$\begin{aligned}\varphi_s \circ \varphi_t(x) &= \varphi_s\big((x-1)e^{-t}+1\big) \\ &= \big((x-1)e^{-t}+1-1\big)e^{-s}+1 \\ &= (x-1)e^{-(t+s)}+1 \\ &= \varphi_{s+t}(x)\end{aligned}$$

(c) (5 points) Find the vector field on $\mathbb{R}$ corresponding to this flow.

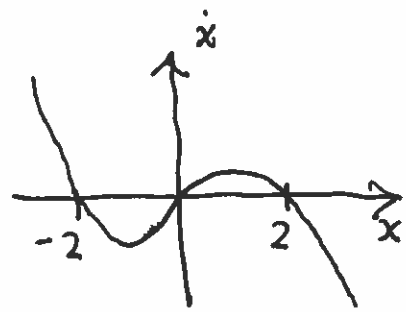
$$\left.\frac{d\varphi_t}{dt}\right|_{t=0} = -(x-1)e^{-t}\Big|_{t=0} = -(x-1).$$

2. (10 points) Consider the following differential equation

$$\dot{x} = -(x-2)(x+2)x.$$

(a) (5 points) **Short Answer:** For the flow corresponding to this differential equation, calculate $\Gamma(1)$.

$\Gamma(1) = \{x \in \mathbb{R} : 0 < x < 2\}$



(b) (5 points) **Short Answer:** For the flow corresponding to this differential equation, calculate $\omega(-3)$.

$\omega(-3) = -2$

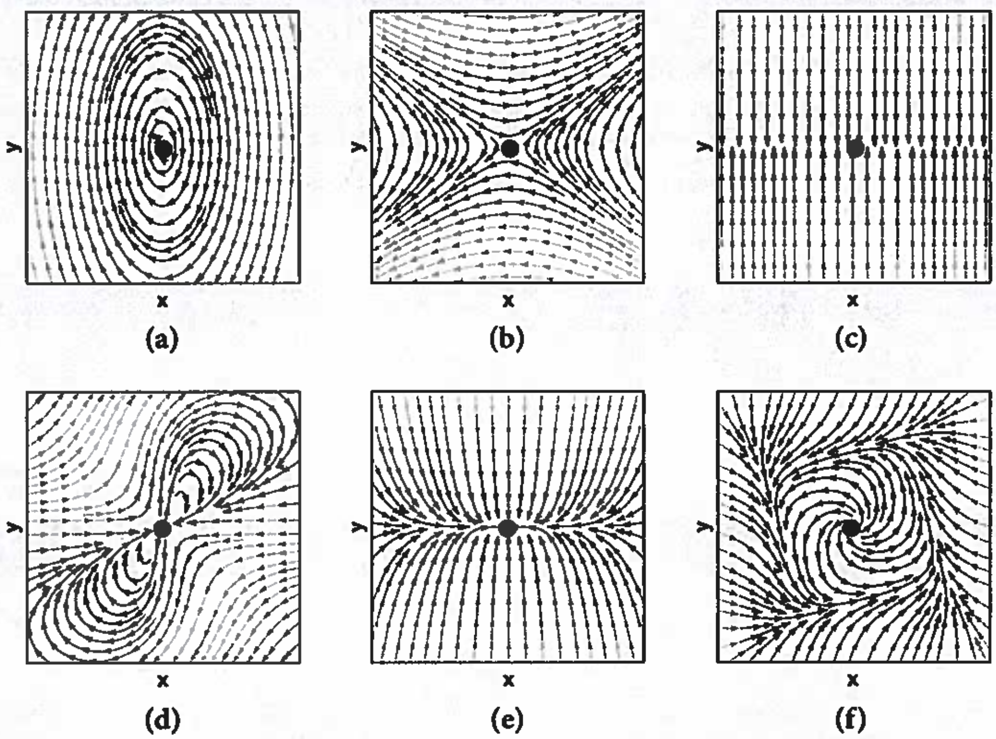
3. (15 points) For a system of differential equations of the form

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y),$$

a fixed point $\mathbf{x}^*$ is **attracting** if there is a neighborhood V of $\mathbf{x}^*$ such that for all $\mathbf{x} \in V$

$$\lim_{t \to \infty} \varphi_t(\mathbf{x}) = \mathbf{x}^*.$$

A fixed points is called **neutrally stable** if it is attracting but not Lyapunov stable. In the below phase portraits the origin is the only fixed point and the qualitative behavior of the phase portrait is the same for all of $\mathbb{R}^2$.



(a) (5 points) **Short Answer:** For which of the phase portraits is the origin Lyapunov stable?

(a), (c), (e)

(b) (5 points) **Short Answer:** For which of the phase portraits is the origin asymptotically stable?

(e)

(c) (5 points) **Short Answer:** For which of the phase portraits is the origin neutrally stable?

(d)

4. (20 points) Consider the operator

$$T[f](x) = 1 + \lambda \int_0^1 x f(y) dy.$$

defined on $C^0([0,1])$ equipped with the sup-norm.

(a) (10 points) Show that if $f \in C^0([0,1])$ then $T[f]$ is a Lipschitz continuous function.

$$T[f](x_2) - T[f](x_1) = \lambda \int_0^1 (x_2 - x_1) f(y) dy$$

$$\Rightarrow |T[f](x_2) - T[f](x_1)| \leq |\lambda| \cdot |x_2 - x_1| \int_0^1 |f(y)| dy$$
$$\leq |\lambda| \, \|f\|_\infty |x_2 - x_1|.$$

(b) (10 points) Find a $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then T is a contraction mapping and if $|\lambda| > \lambda_0$, then it is not.

$$|T[f](x) - T[g](x)| \leq |\lambda| \int_0^1 x |f(x) - g(x)| dx$$
$$\leq |\lambda| \, \|f - g\|_\infty.$$

Therefore, T is a contraction if $|\lambda| < 1$. If we pick $f = 1, g = 0$ we obtain

$$\|T[f] - T[g]\|_\infty = |\lambda|$$

Thus, if $|\lambda| > 1$, T is not a contraction.

5. (15 points) Consider the linear initial value problem problem

$$\dot{x} = A(t)x,$$
$$x(0) = x_0,$$

where $A : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ is an $n \times n$ is a continuous matrix valued function satisfying

$$\|A(t)\|_2 \leq M,$$

for all $t \in \mathbb{R}$ and some $M > 0$.

(a) (5 points) Show that solutions to this differential equation satisfy

$$\|x(t)\|_2 \leq \|x_0\|_2 + \int_0^t M\|x(s)\|_2\, ds.$$

$$x(t) = x_0 + \int_0^t A(t)x(t)\,dt$$

$$\Rightarrow \|x\|_2 \leq \|x_0\|_2 + \int_0^t \|A(t)\|_2 \|x(t)\|_2\,dt$$

$$\leq \|x_0\| + \int_0^t M \|x(t)\|_2\,dt.$$

(b) (10 points) Show that

$$\|x(t)\|_2 \leq \|x_0\|_2 e^{Mt}.$$

Note: You cannot simply state this is true by Gronwall's inequality. I am essentially asking you to show Gronwall's inequality.

Let $G(t) = \|x_0\| + \int_0^t M\|x(t)\|_2\,dt$

$$\Rightarrow G'(t) = M\cdot\|x\|_2 \leq M\cdot G(t)$$

$$\Rightarrow G'(t) - M\cdot G(t) \leq 0$$

$$\Rightarrow \frac{d}{dt}\left(G(t)e^{Mt}\right) \leq 0$$

$$\Rightarrow G(t)e^{-Mt} - G(0) \leq 0$$

$$\Rightarrow \|x\|_2 \leq G(t) \leq G(0)e^{Mt} = \|x_0\|_2 e^{Mt}.$$

6. (20 points) For this problem feel free to use a separate piece of paper. Consider the following system

$$\dot{x} = (1 - x^2)(x - y),$$
$$\dot{y} = x + y(1 - x^2).$$

(a) (5 points) Determine and classify the equilibria in this problem.

Fixed points satisfy $y = x$

$\Rightarrow x(1 + 1 - x^2) = 0$

$\Rightarrow x = 0,\ x = \pm\sqrt{2}$

$$J(x,y) = \begin{bmatrix} -2x(x-y) + (1-x^2) & -(1-x^2) \\ 1 - 2xy & 1 - x^2 \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ Unstable spiral $\quad J(\pm\sqrt{2}, \pm\sqrt{2}) = \begin{bmatrix} -1 & 1 \\ -3 & -1 \end{bmatrix} \Rightarrow$ Stable spiral

(b) (5 points) Show that the lines $x = \pm 1$ are invariant.

If $x = \pm 1$ we have $\dot{x} = 0$ and $\dot{y} = x$. Thus the flow on the line $x = 1$ is given by

$$\varphi_t(\pm 1, y) = (\pm 1, \pm t + y).$$

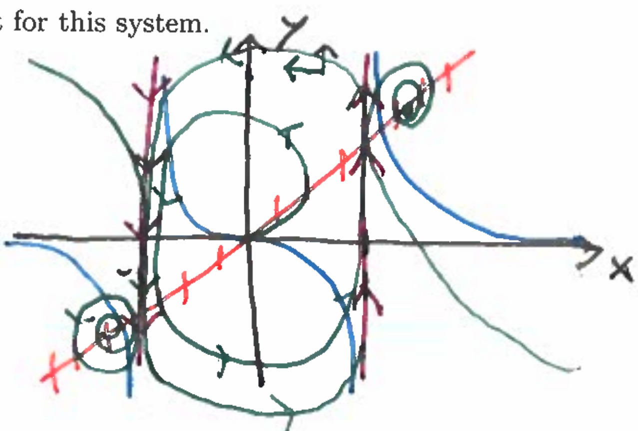
(c) (5 points) Sketch a phase portrait for this system.

<u>Nullclines:</u>

$x = \pm 1$

$y = x$

$y = \dfrac{-x}{1-x^2}$



(d) (5 points) For all values of $(x, y) \in \mathbb{R}^2$, determine the possible ω-limit sets.

- If $x > 1$, $\omega(x,y) = (\sqrt{2}, \sqrt{2})$
- If $-1 < x < 1$, $\omega(x,y) = x = -1 \cup x = 1$
- If $x < -1$, $\omega(x,y) = (-\sqrt{2}, -\sqrt{2})$
- If $x = \pm 1$, $\omega(x,y) = (\pm 1, \pm\infty)$.