MST 750
Homework #1

Due Date: January 14, 2022

1. For the equation $\dot{x} = f(x)$, where f is continuously differentiable, show that if $x(t)$ is a solution then so is $x(t - t_0)$ for any t_0.

2. For the equation $\dot{x} = f(x)$, show that if $\cos(t)$ is a solution, then $-\sin(t)$ is also a solution.

3. For the equation $\ddot{x} = f(x)$, where f is continuously differentiable,

   (a) Show that if $1/(1+t)$ is a solution, then $1/(1-t)$ is also a solution.

   (b) Find f such that $1/(1+t)$ is a solution.

4. Solve the following differential equations:

   (a) $\dot{x} = x^3$

   (b) $\dot{x} = x(1-x)$

   (c) $\dot{x} = x(1-x) - c$

5. Suppose f, g are continuously differentiable functions defined on all of $\mathbb{R}$. Show that the equation
$$\dot{x} = f(x)g(t), \quad x(t_0) = x_0,$$
locally has a unique solution if $f(x_0) \neq 0$. Give an implicit formula for the solution.

6. Solve the following differential equations:

   (a) $\dot{x} = \sin(t)x$

   (b) $\dot{x} = g(t)\tan(x)$

   (c) $\dot{x} = \sin(t)e^x$

7. Consider the following differential equation
$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \geq 0, \\ t\sqrt{|x|}, & x \leq 0. \end{cases}$$

   (a) Show that if $x(t)$ is a solution then so is $-x(t)$ and $x(-t)$.

   (b) Show that without loss of generality, we can assume $x_0 \geq 0$ and $t \geq 0$.

   (c) Show that the initial value problem $x(0) = x_0$ has a unique global solution.

   (d) Show that global solutions can intersect. How does this not violate uniqueness of solutions?