MST 750
Homework #3

Due Date: January 28, 2022

1. Consider a linear system of ordinary differential equations on $\mathbb{R}^n$ defined by

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where $A \in \mathbb{R}^{n\times n}$. In this problem we will prove that the set of solutions forms a linear space of dimension n. To do so we need to prove that that solution set is a vector space and is the span of n linearly independent solutions.

   (a) Show that if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions then $\mathbf{x}_1(t) + \mathbf{x}_2(t)$ and $a\mathbf{x}(t)$ are also solutions for all $a \in \mathbb{R}$. This proves that solutions form a linear subspace of the vector space of continuous curves on $\mathbb{R}^n$ and is thus a vector space.

   (b) Let $e_1, \dots, e_n$ be a basis of $\mathbb{R}^n$. For $i = 1, \dots, n$, let $\mathbf{x}_i(t)$ be the unique solution of this system satisfying $\mathbf{x}_i(0) = e_i$. Assuming existence and uniqueness of solutions, show that all solutions can be written as a linear combination of the functions $\mathbf{x}_i(t)$.

   (c) Show that the functions $\mathbf{x}_i$ are linearly independent.

2. pg. 63, #3.

3. pg. 63, #5.

4. In this problem you will prove that the space of $n \times n$ real valued matrices $\mathbb{R}^{n\times n}$ is a Banach space with the standard matrix norm $\|\cdot\|$. Recall, a Banach space is a complete normed linear space and a complete space is one in which all Cauchy sequences converge to an element of the space. Consequently, all we need to show is that a Cauchy sequence of matrices converges to a real valued matrix.

   (a) Write down the definition of what it means for a sequence of matrices to be a Cauchy sequence with respect to $\|\cdot\|$.

   (b) Prove for all $A \in \mathbb{R}^{n\times n}$ that

$$\max_{j,k} |A_{j,k}| \le \|A\| \le n \max_{j,k} |A_{j,k}| \, .$$

   (c) Use part (b) to prove that if $A^{(n)}$ is a Cauchy sequence with respect to the matrix norm $\|\cdot\|$ then the entries of $A^{(n)}_{i,j}$ are also Cauchy as a sequence of real numbers and thus by completeness of $\mathbb{R}$ converge to a value $A^*_{i,j}$.

   (d) Using part (c) and (b) prove that $A^{(n)}$ converges to A^*, where A^* is the matrix with entries $A^*_{i,j}$.

5. In this problem we will show that for $A \in \mathbb{R}^{n\times n}$, the matrix exponential

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

is well defined.

   (a) Show that if $A, B \in \mathbb{R}^{n\times n}$ then

$$\|AB\| \le \|A\|\|B\|.$$

   Conclude that for all $n \in \mathbb{N}$, $\|A^n\| \le \|A\|^n$. You don't have to be overwrought with showing this conclusion. I don't want to see a trivial induction argument or the use of a compass.

(b) Show that if $A, B \in \mathbb{R}^{n\times n}$ then

$$\|A+B\| \leq \|A\| + \|B\|.$$

Conclude that if $A^{(n)}$ is a sequence in $\mathbb{R}^{n\times n}$ then

$$\left\|\sum_{n=0}^{M} A^{(n)}\right\| \leq \sum_{n=0}^{M} \left\|A^{(n)}\right\|.$$

Again, no need to drag out the proof of the conclusion.

(c) Let $A^{(n)}$ be a sequence in $\mathbb{R}^n$. Show that

$$\sum_{n=0}^{\infty} A^{(n)}$$

converges if $\sum_{n=0}^{\infty} \left\|A^{(n)}\right\|$ converges. **Hint:** The way I like doing problems like this is by showing the sequence of partial sums is Cauchy.

6. Given a matrix $A \in \mathbb{R}^{n\times n}$, let

$$\cos(A) = \frac{e^{iA} + e^{-iA}}{2} \text{ and } \sin(A) = \frac{e^{iA} - e^{-iA}}{2i}.$$

Compute these functions for the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$