MST 750
Homework #3

Due Date: January 28, 2022

1. Consider a linear system of ordinary differential equations on $\mathbb{R}^n$ defined by

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where $A \in \mathbb{R}^{n\times n}$. In this problem we will prove that the set of solutions forms a linear space of dimension n. To do so we need to prove that that solution set is a vector space and is the span of n linearly independent solutions.

   (a) Show that if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions then $\mathbf{x}_1(t) + \mathbf{x}_2(t)$ and $a\mathbf{x}(t)$ are also solutions for all $a \in \mathbb{R}$. This proves that solutions form a linear subspace of the vector space of continuous curves on $\mathbb{R}^n$ and is thus a vector space.

   (b) Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$. For $i = 1, \ldots, n$, let $\mathbf{x}_i(t)$ be the unique solution of this system satisfying $\mathbf{x}_i(0) = e_i$. Assuming existence and uniqueness of solutions, show that all solutions can be written as a linear combination of the functions $\mathbf{x}_i(t)$.

   (c) Show that the functions $\mathbf{x}_i$ are linearly independent.

2. pg. 63, #3.

3. pg. 63, #5.

4. In this problem you will prove that the space of $n \times n$ real valued matrices $\mathbb{R}^{n\times n}$ is a Banach space with the standard matrix norm $\|\cdot\|$. Recall, a Banach space is a complete normed linear space and a complete space is one in which all Cauchy sequences converge to an element of the space. Consequently, all we need to show is that a Cauchy sequence of matrices converges to a real valued matrix.

   (a) Write down the definition of what it means for a sequence of matrices to be a Cauchy sequence with respect to $\|\cdot\|$.

   (b) Prove for all $A \in \mathbb{R}^{n\times n}$ that

$$\max_{j,k} |A_{j,k}| \le \|A\| \le n \max_{j,k} |A_{j,k}| .$$

   (c) Use part (b) to prove that if $A^{(n)}$ is a Cauchy sequence with respect to the matrix norm $\|\cdot\|$ then the entries of $A^{(n)}_{i,j}$ are also Cauchy as a sequence of real numbers and thus by completeness of $\mathbb{R}$ converge to a value $A^*_{i,j}$.

   (d) Using part (c) and (b) prove that $A^{(n)}$ converges to A^*, where A^* is the matrix with entries $A^*_{i,j}$.

5. In this problem we will show that for $A \in \mathbb{R}^{n\times n}$, the matrix exponential

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

is well defined.

   (a) Show that if $A, B \in \mathbb{R}^{n\times n}$ then

$$\|AB\| \le \|A\|\|B\|.$$

   Conclude that for all $n \in \mathbb{N}$, $\|A^n\| \le \|A\|^n$. You don't have to be overwrought with showing this conclusion. I don't want to see a trivial induction argument or the use of a compass.

(b) Show that if $A, B \in \mathbb{R}^{n\times n}$ then

$$\|A+B\| \leq \|A\| + \|B\|.$$

Conclude that if $A^{(n)}$ is a sequence in $\mathbb{R}^{n\times n}$ then

$$\left\|\sum_{n=0}^{M} A^{(n)}\right\| \leq \sum_{n=0}^{M} \left\|A^{(n)}\right\|.$$

Again, no need to drag out the proof of the conclusion.

(c) Let $A^{(n)}$ be a sequence in $\mathbb{R}^n$. Show that

$$\sum_{n=0}^{\infty} A^{(n)}$$

converges if $\sum_{n=0}^{\infty} \|A^{(n)}\|$ converges. **Hint:** The way I like doing problems like this is by showing the sequence of partial sums is Cauchy.

6. Given a matrix $A \in \mathbb{R}^{n\times n}$, let

$$\cos(A) = \frac{e^{iA} + e^{-iA}}{2} \text{ and } \sin(A) = \frac{e^{iA} - e^{-iA}}{2i}.$$

Compute these functions for the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Homework #3

#1.

Prove that the set of solutions to the equation

$$\vec{x}' = A\vec{x}$$

forms a linear subspace of dimension n.

Solution:

(a) First, if $\vec{x}_1, \vec{x}_2$ are solutions and $a \in \mathbb{R}$ it follows that if $\vec{y}_1 = \vec{x}_1 + \vec{x}_2$ and $\vec{y}_2 = a\vec{x}_1$ then

$$\begin{aligned} \dot{\vec{y}}_1 &= \dot{\vec{x}}_1 + \dot{\vec{x}}_2 \\ &= A\vec{x}_1 + A\vec{x}_2 \\ &= A(\vec{x}_1 + \vec{x}_2) \\ &= A\vec{y}_1 \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{y}}_2 &= a\dot{\vec{x}}_1 \\ &= aA\vec{x}_1 \\ &= Aa\vec{x}_1 \\ &= A\vec{y}_2 . \end{aligned}$$

That is, $\vec{y}_1, \vec{y}_2$ are solutions as well.

(b) Let $\vec{x}_i(t)$ be the unique solution satisfying

$$\vec{x}_i(0) = \vec{x}_0 = c_1\vec{e}_1 + \dots + c_n\vec{e}_n.$$

Therefore, by existence and uniqueness of solutions:

$$\vec{x}(t) = c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t).$$

(c). Finally, suppose there exists $c_1, \dots, c_n \in \mathbb{R}^n$ such that

$$c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t) = 0.$$

By existence and uniqueness, the above equation must be true for all t since $\vec{x}(t) = \vec{0}$ is a solution. Consequently,

$$c_1\vec{x}_1(0) + \dots + c_n\vec{x}_n(0) = 0$$

$$\Rightarrow c_1\vec{e}_1 + \dots + c_n\vec{e}_n = 0$$

Therefore, by linear independence of $\vec{z}_1, \ldots, \vec{z}_n$ it follows that $c_1 = \ldots = c_n = 0$. Consequently, $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent.

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#2.

Show that if is T is a bounded linear operator and is invertible, then

$$\|T^{-1}\| \geq \frac{1}{\|T\|}$$

~~proof~~:

$$1 = \|I\| = \|T^{-1}T\| \leq \|T^{-1}\| \cdot \|T\|$$

$$\Rightarrow \quad \|T\| \geq \frac{1}{\|T^{-1}\|}$$

■

#3.

Prove that a linear operator is bounded if and only if it is continuous.

~~proof~~:

(a) We first prove that if T is continuous at 0 it is continuous everywhere. If T is continuous at 0 it follows that for all $x_n \to 0$ that:

$$\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(0) = 0.$$

Now, suppose $y_n \to y$ and define $x_n = y_n - y$. Consequently, $x_n \to 0$ and thus by linearity:

$$\lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} T(x_n + y)$$

$$= \lim_{n\to\infty} T(x_n) + \lim_{n\to\infty} T(y)$$

$$= T(\lim_{n\to\infty} x_n) + T(y)$$

$$= T(0) + T(\lim_{n\to\infty} y_n)$$

Therefore,
$$\lim_{n\to\infty} T(y_n) = T(\lim_{n\to\infty} y_n).$$

(b) Now, if $x_n \to 0$ and T is bounded it follows that
$$\|T(x_n)\| \leq \|T\| \cdot \|x_n\|$$
and thus if $x_n \to 0$ it follows that $\|T(x_n)\| \to 0$.

(c) Suppose T is unbounded. Therefore, there exists a sequence x_n such that $\|T(x_n)\| \geq n\|x_n\|$. Letting $y_n = x_n/(n\|x_n\|)$ it follows that $\|y_n\| = 1/n$ and thus $y_n \to 0$. However,
$$\|T(y_n)\| = \frac{\|T(x_n)\|}{n\|x_n\|} \geq 1$$
and thus $T(y_n) \not\to 0$ proving T is not continuous at 0. Therefore, if T is continuous at 0 it is bounded.

$\blacksquare$.

#4.

Prove that $\mathbb{R}^{n\times n}$ is a Banach space with the induced norm.

proof:

Let $A^{(m)} \in \mathbb{R}^{n\times n}$ be a Cauchy sequence with respect to the induced norm. That is for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, K \geq N$ implies $\|A^{(m)} - A^{(K)}\| < \varepsilon$. To prove $A^{(m)}$ converges we first construct a candidate limit point. To do so we prove two intermediate results.

1. First, note that
$$\begin{aligned} \|A(x)\| &= \|\sum A_{ij} x_j\| \\ &= \Big(\sum_i \big(\sum_j A_{ij} x_j\big)^2\Big)^{1/2} \\ &\leq \Big(\sum_i \big(\sum_j |A_{ij}| \cdot |x_j|\big)^2\Big)^{1/2} \\ &\leq \Big(\sum_i \big(\sum_j \max_{i,j} |A_{ij}| \cdot |x_j|\big)^2\Big)^{1/2}. \end{aligned}$$

Consequently,
$$\|A(x)\| \leq \max_{i,j} |A_{ij}| \left(\sum_i \left(\sum_j |x_j|\right)^2\right)^{1/2}.$$

If $\|x\|=1$ it follows that $|x_j| \leq 1$ and thus
$$\|A(x)\| \leq \max_{i,j} |A_{ij}| \left(\sum_i \sum_j 1\right)^{1/2}$$
$$= \max_{i,j} |A_{ij}|\, n.$$

Therefore,
$$\|A\| = \sup_{\|x\|=1} \|A(x)\| \leq n \max_{i,j} |A_{ij}|.$$

2. Let i^*, j^* satisfy $\max_{i,j} |A_{ij}| = |A_{i^*j^*}|$. Let $\vec{x}^*$ satisfy
$$x_i^* = \begin{cases} 0 & \text{if } i \neq j^* \\ 1 & \text{if } i = j^* \end{cases}.$$

Therefore,
$$\|A(\vec{x}^*)\| = \left(\sum_i \left(\sum_j A_{ij} x_j\right)^2\right)^{1/2}$$
$$= \left(\sum_i A_{ij^*}{}^2\right)^{1/2}$$
$$\geq \left(A_{i^*j^*}{}^2\right)^{1/2}$$
$$= |A_{i^*j^*}|.$$

Consequently, since $\|\vec{x}^*\|=1$ it follows that
$$\|A\| = \sup_{\|x\|=1} \|A(x)\| \geq \|A(\vec{x}^*)\| \geq |A_{i^*j^*}| = \max_{i,j} |A_{ij}|.$$

By items 1-2 it follows that
$$\max_{i,j} |A_{ij}| \leq \|A\| \leq n \max_{i,j} |A_{ij}|.$$

Consequently, for all i,j $|A_{ij}| \leq \|A\|$. It follows that if A is Cauchy then so are the real numbers A_{ij} since
$$|A_{ij}^{(m)} - A_{ij}^{(k)}| \leq \|A_{ij}^{(m)} - A_{ij}^{(k)}\|.$$

Therefore, for each i,j there exists A_{ij} such that $A_{ij}^{(m)} \to A_{ij}$. Let $A \in \mathbb{R}^{n\times n}$ be the matrix with entries A_{ij}. Since, $\max_{i,j} |A_{ij}^{(m)} - A_{ij}| \to 0$ it follows that $\|A^{(m)} - A\| \to 0$ since

$$\|A^{(m)} - A\| < \max_{i,j} |A_{ij}^{(m)} - A_{ij}|.$$

$\blacksquare$

#5.

Prove that for $A \in \mathbb{R}^{n\times n}$,

$$\exp(A) = \sum_{n\geq 0}^{\infty} \frac{1}{n!} A^n$$

is well defined.

~~proof~~:

For $A, B \in \mathbb{R}^{n\times n}$ it follows that since $\|Ay\| \leq \|A\| \|y\|$ that

$$\|AB\| = \sup_x \|ABx\|$$

$$\leq \sup_x \|A\| \cdot \|Bx\|$$

$$= \|A\| \cdot \|B\|.$$

Consequently, for all $k \in \mathbb{N}$

$$\|A^k\| \leq \|A\|^k.$$

Moreover,

$$\|A+B\| = \sup_x \|(A+B)x\| \leq \sup_x \|Ax + Bx\| \leq \sup_x \|Ax\| + \|Bx\|$$

$$\Rightarrow \|A+B\| \leq \sup_x \|Ax\| + \sup_x \|Bx\| = \|A\| + \|B\|.$$

If we let

$$S^{(m)} = \sum_{n\geq 0}^{m} 1/n!\, A^n$$

it follows that

$$\|S^{(k)} - S^{(m)}\| = \|\sum_{n\geq m}^{k} 1/n!\, A^n\|$$

$$\leq \sum_{n\geq m}^{k} 1/n!\, \|A\|^n = |s_k - s_m|,$$

Where s_K is the sequence of real numbers defined by:

$$s_K = \sum_{n=0}^{K} \frac{1}{n!} \|A\|^n.$$

Since $\lim_{K \to \infty} s_K = \exp(\|A\|)$ it follows that s_K is Cauchy and thus S_K is Cauchy as well and thus by completeness convergent. ∎