MST 750
Homework #6

Due Date: March 18, 2022

1. Consider the the following differential equation
$$\dot{x} = f(x,t),$$
where $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is continuous. Show that if $|f(t,x) - f(t,y)| \leq L(t)|x-y|$ then
$$|x(t) - y(t)| \leq |x_0 - y_0| \exp\left(\left|\int_{t_0}^{t} L(s)ds\right|\right),$$
where x, y are solutions to the ordinary differential equation satisfying $x(t_0) = x_0$, $y(t_0) = y_0$.

2. Let $u, v, w \in C^0([a,b];\mathbb{R})$ with $w > 0$ such that
$$u(t) \leq v(t) + \int_a^t w(s)u(s)\,ds$$
for every $t \in [a,b]$. Prove that
$$u(t) \leq v(t) + \int_a^t w(s)v(s)\exp\left(\int_s^t w(u)\,du\right) ds.$$

3. pg. 153, #1

4. pg. 153, #2

5. pg. 153, #3

6. pg. 153, #4

7. Consider the following differential equation
$$\dot{x} = f(x),$$
where $f : \mathbb{R} \mapsto \mathbb{R}$ is a differentiable function satisfying $f(0) = f(1) = 0$ and $f(x) > 0$ for $x \in (0,1)$. Determine $\Gamma(x)$ and $\omega(x)$ if $x \in [0,1]$.

8. Denote by $d(x,A) = \inf_{y \in A} |x-y|$ the distance between a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$.
   (a) Show that $|d(x,A) - d(z,A)| \leq |x-z|$.
   (b) Prove that the mapping $x \mapsto d(x,A)$ is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$.

9. For a function $g \in C^2(\mathbb{R}^2;\mathbb{R})$, consider the equation
$$\dot{x} = -\nabla g(x).$$
   (a) Show that if u is a nonconstant solution, then $g \circ u$ is strictly decreasing.
   (b) Show this system has no periodic orbits.
   (c) For the function $g(x,y) = x^2y^4$ sketch the level sets of $g(x,y)$ overlaid on top of a phase portrait. What geometric condition must hold between the level sets and the orbits?

10. Consider the following equation in polar coordinates:
$$\dot{r} = f(r),$$
$$\dot{\theta} = 1,$$
where
$$f(r) = \begin{cases} r\sin(1/r^2), & r \neq 0 \\ 0, & r = 0 \end{cases}.$$
Show that the origin is Lyapunov stable but not asymptotically stable.

Homework #6

#1.

Consider the following differential equation

$$\dot{x} = f(x,t) \quad (*)$$

Where $f: \mathbb{R}^n \times \mathbb{R}$ is continuous. Show that if

$$|f(t,x) - f(t,y)| \leq L(t)|x-y|$$

then

$$|x(t) - y(t)| \leq |x_0 - y_0| \exp\left(\left|\int_{t_0}^{t} L(s)\,ds\right|\right),$$

where x, y are solutions to $(*)$ satisfying

$$x(t_0) = x_0, \; y(t_0) = y_0.$$

Solution:

Let $E(t) = |x(t) - y(t)|^2$. Differentiating it follows that

$$\frac{dE}{dt} = 2\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle$$

$$= 2\langle x(t) - y(t), f(x(t),t) - f(y(t),t)\rangle$$

$$\leq 2\|x(t) - y(t)\| \cdot \|f(x(t),t) - f(y(t),t)\|$$

$$\leq 2L(t)\|x(t) - y(t)\|^2$$

$$= 2L(t)E(t).$$

Therefore,

$$\frac{dE}{dt} - 2L(t)E(t) \leq 0$$

$$\Rightarrow \frac{d}{dt}\left(e^{-2\int_{t_0}^{t} L(t)} E(t)\right) \leq 0$$

$$\Rightarrow e^{-2\int_{t_0}^{t} L(t)} E(t) - E(t_0) \leq 0$$

$$\Rightarrow E(t) \leq e^{2\int_{t_0}^{t} L(t)} E(t_0)$$

$$\Rightarrow |x(t) - y(t)|^2 \leq e^{2\int_{t_0}^{t} L(t)\,dt} |x_0 - y_0|^2$$

$$\Rightarrow |x(t) - y(t)| \leq e^{\int_{t_0}^{t} L(t)\,dt} |x_0 - y_0|. \quad \blacksquare$$

#2

Let $u, v, w \in \mathcal{C}^0([a,b]; \mathbb{R})$ with $w > 0$ such that

$$u(t) \le v(t) + \int_a^t w(s)u(s)\,ds$$

for every $t \in [a,b]$. Prove that

$$u(t) \le v(t) + \int_a^t w(s)v(s)\exp\left(\int_s^t w(u)\,du\right)ds.$$

~~proof~~:

Let $G(t) = v(t) + \int_a^t w(s)u(s)\,ds$. Differentiating, it follows that

$$G'(t) = v'(t) + w(t)u(t) \le v'(t) + w(t)G(t).$$

Therefore,

$$\frac{d}{dt}\left(e^{-\int_a^t w(s)ds}G(t)\right) \le v'(t)e^{-\int_a^t w(s)ds}.$$

Integrating both sides and integrating the right hand side by parts we have that:

$$\int_{G(a)}^{\exp(-\int_a^t w(s)ds)G(t)} d\left(\exp\left(-\int_a^t w(s)ds\, G(t)\right)\right) \le \int_a^t v'(u)e^{-\int_a^u w(s)ds}du$$

$$\Rightarrow \exp\left(-\int_a^t w(s)ds\right)G(t) - v(a) \le v(t)e^{-\int_a^u w(s)ds} - v(a) + \int_a^t v(u)w(u)e^{-\int_a^u w(s)}$$

$$\Rightarrow G(t) \le v(t) + e^{\int_a^t w(s)ds}\int_a^t v(u)w(u)e^{-\int_a^u w(s)ds}du$$

$$= \int_a^t v(u)w(u)\exp\left(\int_a^t w(s)ds - \int_a^u w(s)ds\right)du$$

Since $u \le t$ in the innermost integral it follows that

$$\int_a^t w(s)ds - \int_a^u w(s)ds = \int_u^t w(s)ds.$$

Therefore,

$$u(t) \le G(t) \le \int_a^t v(u)w(u)\exp\left(\int_u^t w(s)ds\right)du$$

$$= \int_a^t v(s)w(s)\exp\left(\int_s^t w(u)du\right)ds.$$

■

#4.

Sketch phase portraits for the following:

(a) $\dot{x} = y$

$\dot{y} = x - x^3 - ay$

(b) $\dot{x} = x^2 - y^2 - 1$

$\dot{y} = 2y$

(c) $\dot{x} = y - x^2 + 2$

$\dot{y} = 2y^2 - 2xy$

(d) $\dot{x} = -4x - 2y + 4$

$\dot{y} = xy$

Solution:

(a). Nullclines:

$y = 0 \quad (\dot{x} = 0)$

$y = \frac{1}{a}(x - x^3) \quad (\dot{y} = 0)$

Fixed Points:

$(0,0), (1,0), (-1,0)$

Jacobian:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & -a \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix} \qquad J(1,0) = \begin{bmatrix} 0 & 1 \\ -2 & -a \end{bmatrix}$$

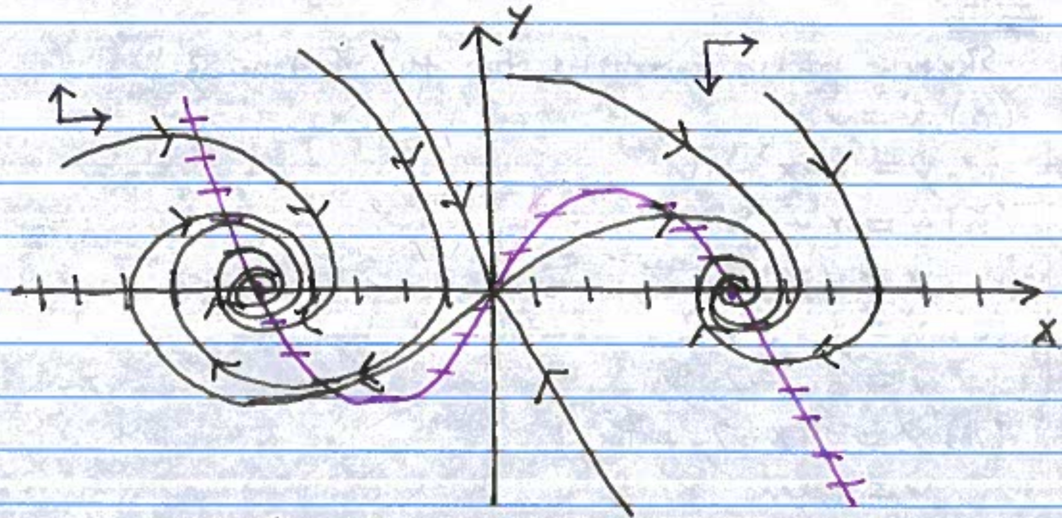
$\lambda_1 + \lambda_2 = -a \qquad\qquad \lambda_1 + \lambda_2 = -a$

$\lambda_1 \lambda_2 = -1 \qquad\qquad \lambda_1 \lambda_2 = 2$

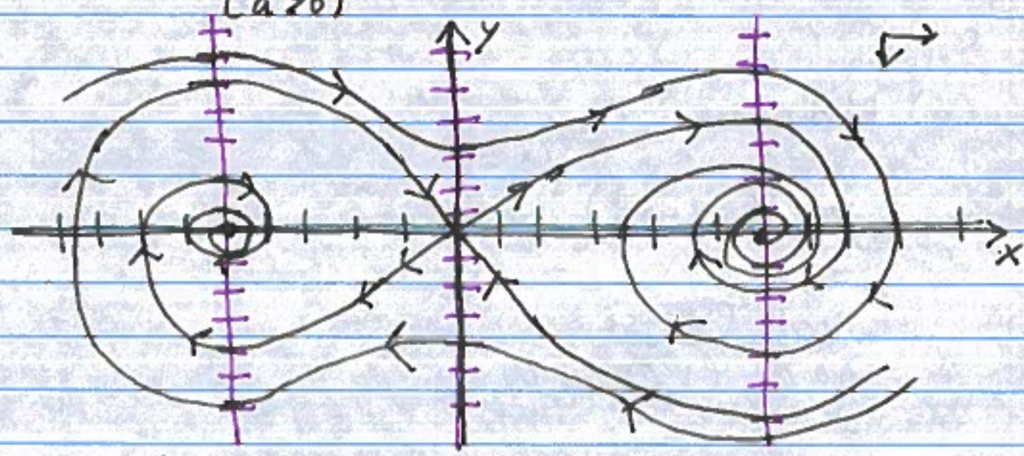
$$\lambda = \frac{-a \pm \sqrt{a^2+4}}{2} \qquad \lambda = \frac{-a \pm \sqrt{a^2-4}}{2}$$

(Saddle) (Stable Node, possibly a spiral)

$$J(-1,0) = \begin{bmatrix} 0 & 1 \\ -2 & -a \end{bmatrix} \Rightarrow \text{same classification as } (1,0).$$



(a>0)



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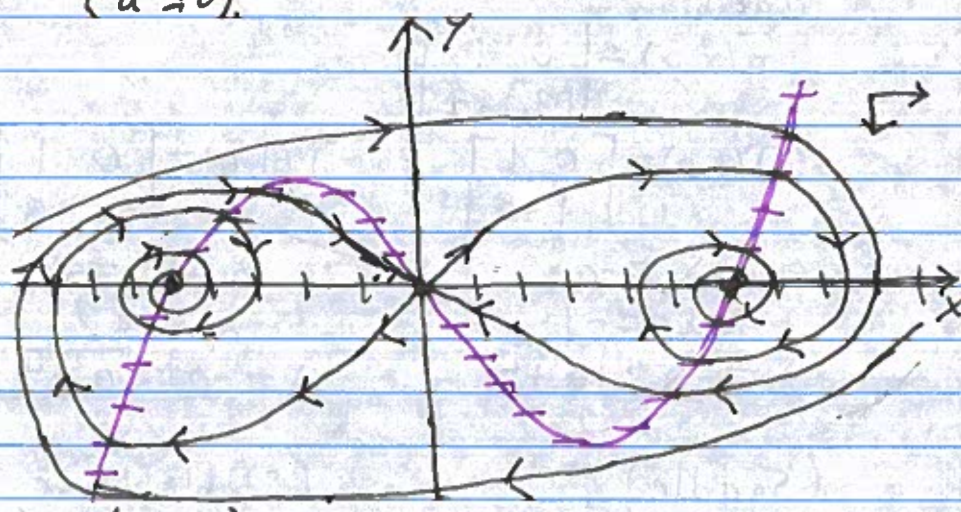


(a<0)

#9

For a function $g \in C^2(\mathbb{R}^2; \mathbb{R})$ consider the equation

$$\dot{x} = -\nabla g(x).$$

(a) Show that if u is a nonconstant solution, then $g \circ u$ is strictly decreasing.

(b) Show this system has no periodic orbits.

(c) For the function $g(x,y) = x^2y^4$ sketch the level sets of $g(x,y)$ overlaid on top of a phase portrait. What geometric condition must hold between the level sets and the orbits?

Solution:

(a) $\frac{d}{dt} g(x(t)) = \nabla g \cdot \frac{dx}{dt} = -\nabla g \cdot \nabla g = -|\nabla g|^2 < 0.$

(b) Suppose u^* is a periodic orbit of period T. Therefore,

$$g(u^*(t_0)) = g(u^*(t_0+T)),$$

which contradicts the fact that $g(u(t))$ is decreasing.

(c). If $g(x,y) = x^2y^4$ then

$$\nabla g = (2xy^4, 4x^2y^3)$$

$$\Rightarrow \dot{x} = -2xy^4,$$

$$\dot{y} = -4x^2y^3,$$

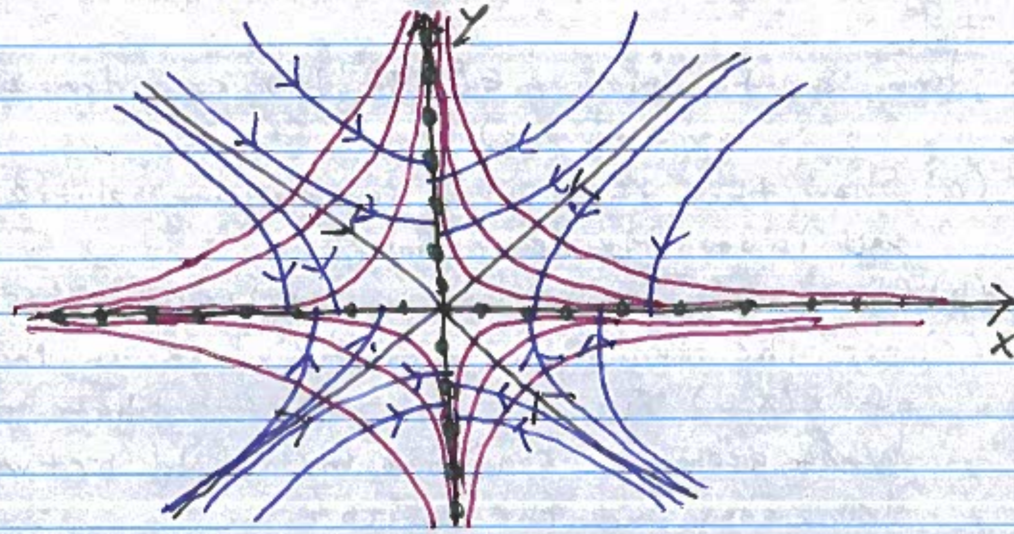
The level sets of g are orthogonal to the orbits of this dynamical system. The level sets are given by:

$$x^2y^4 = c$$

for $c \geq 0$. Therefore,

$$y^2 = \frac{c^{1/2}}{|x|}$$

$$\Rightarrow y = \pm \frac{c^{1/4}}{\sqrt{|x|}}.$$



#10.

Consider the following equation in polar coordinates

$$\dot{r} = f(r),$$
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where

$$f(r) = \begin{cases} r\sin(1/r^2), & r \neq 0. \\ 0 & r = 0 \end{cases}$$

Show that the origin is Lyapunov stable but not asymptotically stable.

Solution:

The zeros for $f(r)$ are given by

$$r_n^* = 1/\sqrt{n\pi},$$

where $n \in \mathbb{N}$. Consequently, for all $r_0 \in \mathbb{R}^+$ there exists $n_{r_0} \in \mathbb{N}$ such that for all $t \in \mathbb{R}$

$$r^*_{n_{r_0}+1} < r(t) < r^*_{n_{r_0}}.$$

Consequently, 0 is Lyapunov stable.