MST 750
Homework #7

Due Date: March 25, 2022

1. Can $\phi(t) = (\sin(t), \sin(2t))$ be the solution of an autonmous system $\dot{x} = f(x)$?
2. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a function of class C^∞ such that the equation $\dot{x} = f(x)$ defines a flow φ_t in $\mathbb{R}^n$.
   (a) Show that
   $$\varphi_t(x) = x + f(x)t + \frac{1}{2}\nabla f(x) f(x) t^2 + o(t^2).$$
   (b) Verify that
   $$\det(\nabla\varphi_t(x)) = 1 + \nabla \cdot f(x)t + o(t).$$
   (c) Given an open set $A \subset \mathbb{R}^n$ and $t \in \mathbb{R}$ show that
   $$\frac{d}{dt}\mu(\varphi_t(A)) = \int_{\varphi_t(A)} \nabla \cdot f(x) dV,$$
   where μ denotes the volume of a set in $\mathbb{R}^n$. **Hint:** First argue that
   $$\frac{d}{dt}\mu(\varphi_t(A)) = \int_{\partial\varphi_t(A)} f(x) \cdot \mathbf{n} dA,$$
   where $\partial\varphi_t(A)$ denotes the boundary of $\varphi_t(A)$ with outward normal $\mathbf{n}$. You do not have to do this argument rigorously and a version of this argument can be found in Strogatz.
   (d) Show that if $\nabla \cdot f(x) = 0$, then the equation $\dot{x} = f(x)$ has neither asymptotically stable fixed points nor asymptotically stable periodic solutions.
3. pg. 156, #12
4. pg. 156, #13
5. pg. 156, #14
6. pg. 157, #17
7. pg. 157, #18

Homework #7:

#2.

(a) Taylor expanding we have that

$$\varphi_t(x) = \varphi_0(x) + \frac{d\varphi_t}{dx}\Big|_{t=0} t + \frac{1}{2}\frac{d^2\varphi}{dt^2}\Big|_{t=0} t^2 + o(t^3)$$

$$= x + f(x)t + \frac{1}{2}\frac{d}{dt}(f(x))\, t^2 + o(t^3)$$

$$= x + f(x)t + \frac{1}{2}\nabla f \cdot f\, t^2 + o(t^3).$$

(b) By part (a) we have that

$$\nabla\varphi_t(x) = I + \nabla f(x)\, t + o(t)$$

$$\Rightarrow \mathrm{Det}(\nabla\varphi_t(x)) = \mathrm{Det}(I + \nabla f(x)t + o(t))$$

$$= \mathrm{Det}\left(\delta_{ij} + \frac{\partial f_i}{\partial x_j} t + o(t)\right)$$

$$= \mathrm{Det}\begin{pmatrix} 1 + \frac{\partial f_1}{\partial x_1}t & \cdots & \frac{\partial f_1}{\partial x_n}t \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & 1 + \frac{\partial f_n}{\partial x_n}t \end{pmatrix} + o(t)$$

$$= 1 + \left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}\right) t + o(t)$$

$$= 1 + \nabla \cdot f t + o(t)$$

(c) The volume swept out in a small interval of time Δt is given by:

$$\mu(\varphi_{t+\Delta t}(A)) - \mu(\varphi_t(A)) = \left(\int_{\partial\varphi_t(A)} f \cdot n\, dA\right)\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{d}{dt}\mu(\varphi_t(A)) = \int_{\partial\varphi_t(A)} f(x)\cdot\vec{n}\, dA.$$

(d). Since fixed points and limit cycles have zero volume they cannot be the limiting flow of balls in $\mathbb{R}^n$ if $\nabla \cdot f = 0$. Consequently, they cannot be asymptotically stable.

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#3

Assume $\varphi_t : A \to A$ is conjugate to $\psi_t : B \to B$ with conjugacy $h : A \to B$.

(a) Show that if $\omega(x)$ is the limit set for $x \in A$ under φ, then $h(\omega(h^{-1}(y)))$ is the ω-limit set for $y = h(x)$ under ψ.

(b) Show that if Λ is invariant for φ, then $h(\Lambda)$ is invariant for ψ.

(c) Show that if $W^s(\Lambda)$ is the basin for Λ, then $h(W^s(\Lambda))$ is the basin of $h(\Lambda)$.

(d) Show that if Λ is an attractor, then so is $h(\Lambda)$.

Solution:

(a) First by construction

$$h(\omega(h^{-1}(y)) = h(\omega(x)).$$

Consequently, $y \in h(\omega(h^{-1}(y))$ if and only if $h^{-1}(y) \in \omega(x)$.

Now, $y^* \in \omega(h(x))$ if and only if there exists a subsequence t_k such that

$$\lim_{k \to \infty} \psi_{t_k}(h(x)) = y^*$$

$$\Rightarrow \lim_{k \to \infty} h(\varphi_{t_k}(x)) = y^*$$

$$\Rightarrow h(\lim_{k \to \infty} \varphi_{t_k}(x)) = y^*$$

$$\Rightarrow \lim_{k \to \infty} \varphi_{t_k}(x) = h^{-1}(y^*)$$

Consequently, $h^{-1}(y^*) \in \omega(x)$.

(b). Let $y \in h(\Lambda)$. Therefore,

$$\psi_t(h(\Lambda)) = h(\varphi_t(\Lambda))$$
$$= h(\Lambda).$$

(c) Let $y \in h(W^s(\Lambda))$. Then,

$$\lim_{t\to\infty} \rho(\psi_t(y), h(\Lambda)) = \lim_{t\to\infty} \rho(\psi_t(h(x)), h(\Lambda))$$
$$= \lim_{t\to\infty} \rho(h(\varphi_t(x)), h(\Lambda))$$
$$= \rho(\lim_{t\to\infty} h(\varphi_t(x)), h(\Lambda))$$
$$= \rho(h(\lim_{t\to\infty} \varphi_t(x)), h(\Lambda))$$
$$= 0$$

Since $\lim_{t\to\infty} \varphi_t(x) \in \Lambda$. All of the arguments can be reversed to obtain the apposite direction.

(d). Follows from above parts.

#4.

Assume φ and ψ are flows on $\mathbb{R}^2$ that have exactly two equilibria that are both saddles. Suppose for the flow φ that the unstable set of one saddle corresponds to the stable set of the other but this is not true for ψ. Show that φ and ψ are not topologically conjugate.

Solution:

Let x_1, x_2 denote saddles for φ and suppose

$$W^u(x_1) = W^s(x_2).$$

Therefore, $\omega(W^u(x_1)) = x_2$. If we let h be the corresponding homeomorphism between φ and ψ it follows that $h(x_1)$ and $h(x_2)$ are fixed points of ψ and by problem #3

$$\omega(h(W^u(x_1))) = h(x_2)$$

And thus $h(W^u(x_1)) = W^s(h(x_2))$. A similar argument with α-limit set shows that

$$\alpha(h(W^u(x_1)) = h(x_1)$$

and thus

$$W^s(h(x_1)) = h(W^u(x_1))$$

which is a contradiction.

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#5

Show that if $y \in \omega(x)$, then y is nonwandering.

Solution:

For contradiction suppose $y \in \omega(x)$ is wandering. Therefore, there exists a nbd W of y and a time $T > 0$ such that $\varphi_t(W) \cap W = \emptyset$. However, since $y \in \omega(x)$ there exists a subsequence t_k such that $\varphi_{t_k}(y) \to y$ which implies there exists T' such that $t_k > T'$ implies $\varphi_{t_k}(y) \in W$ which is a contradiction.

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#7

$$\dot{x} = y$$
$$\dot{y} = x - \alpha y - xz$$
$$\dot{z} = -\beta z + x^2.$$

Solution:

The fixed points satisfy

$$y = 0$$
$$\Rightarrow x(1-z) = 0$$
$$\Rightarrow z = 1 \text{ or } x = 0$$

$$\Rightarrow x^2 = \pm\sqrt{\beta} \text{ or } z = 0.$$

The three fixed points are

$(\pm\sqrt{\beta}, 0, 1)$ and $(0, 0, 0)$.

Therefore, the Jacobian is given by

$$J(x,y,z) = \begin{bmatrix} 0 & 1 & 0 \\ 1-z & -\alpha & -x \\ 2x & 0 & -\beta \end{bmatrix}$$

$$\Rightarrow J(0,0,0) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & -\beta \end{bmatrix}.$$

$J(0,0,0)$ is block diagonal and thus has eigenvalues

$$\lambda_1 = -\beta$$

$$\lambda_{2,3} = \frac{-\alpha \pm \sqrt{\alpha^2+4}}{2}$$

and is thus always unstable.

$$J(\pm\sqrt{\beta},0,1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & \mp\sqrt{\beta} \\ \pm 2\sqrt{\beta} & 0 & -\beta \end{bmatrix}$$

$$\Rightarrow \lambda I - J(\pm\sqrt{\beta},0,1) = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda+\alpha & \pm\sqrt{\beta} \\ \mp 2\sqrt{\beta} & 0 & \lambda+\beta \end{bmatrix}$$

The characteristic polynomial is therefore:

$$\lambda(\lambda+\alpha)(\lambda+\beta) - 2\beta = p(\lambda)$$

$$\Rightarrow p(\lambda) = \lambda^3 + (\alpha+\beta)\lambda^2 + \alpha\beta\lambda + 2\beta$$

$$\Rightarrow p'(\lambda) = 3\lambda^2 + 2(\alpha+\beta)\lambda + \alpha\beta$$

The critical points of $p(\lambda)$ are given by

$$\lambda^*_\pm = \frac{-2(\alpha+\beta) \pm \sqrt{4(\alpha+\beta)^2 - 12\alpha\beta}}{6}$$

$$= \frac{-2(\alpha+\beta) \pm 2\sqrt{\alpha^2-\alpha\beta+\beta^2}}{6}$$

$$= \frac{-(\alpha+\beta) \pm \sqrt{\alpha^2-\alpha\beta+\beta^2}}{3}.$$

These critical points exist if $\alpha^2-\alpha\beta+\beta^2>0 \Rightarrow \alpha^2+\beta^2>\alpha\beta$. This is trivially true if $\alpha\beta<0$. If $\alpha\beta>0$ then

$$\alpha^2-\alpha\beta+\beta^2 = \alpha^2-2\alpha\beta+\beta^2+\alpha\beta = (\alpha+\beta)^2+\alpha\beta>0.$$

Since $p(0) = 2\beta$ and $\lim_{\lambda \to \infty} p(\lambda) = \infty$ to ensure stability we need $p(\lambda_+) > 0$.