

MST 750  
Homework #8

Due Date: April 8, 2022

1. pg. 369, #1
2. pg. 369, #2. (By discuss the phase portrait he means sketch every possible phase portrait).

## Homework #8

#1.

Let  $x_i \in \mathbb{R}^3$  and consider the system

$$m_i \ddot{x}_i = \sum_{j \neq i} f(x_j - x_i)$$

Where  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the force.

(a) Show that total momentum is conserved if  $f$  is odd.

(b) Show that angular momentum,  $L = \sum_i m_i \dot{x}_i \times x_i$  is an invariant if  $f(x) = xg(|x|)$ .

Solution:

$$\begin{aligned} \text{(a) } \frac{dP}{dt} &= \sum_{i=1}^N m_i \ddot{x}_i = \sum_{i=1}^N \sum_{j \neq i} f(x_j - x_i) \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} f(x_j - x_i) + f(x_i - x_j) \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} (f(x_j - x_i) - f(x_i - x_j)) \\ &= 0. \end{aligned}$$

$$\begin{aligned} \text{(b) } \frac{dL}{dt} &= \sum_i m_i \ddot{x}_i \times x_i + \sum_i m_i \dot{x}_i \times \dot{x}_i \\ &= \sum_i \sum_{j \neq i} (x_j - x_i) g(|x_j - x_i|) \times x_i \\ &= \sum_i \sum_{j \neq i} x_j \times x_i g(|x_j - x_i|) \\ &= \frac{1}{2} \sum_i \sum_{j \neq i} (x_j \times x_i g(|x_j - x_i|) + x_i \times x_j g(|x_i - x_j|)) \\ &= \frac{1}{2} \sum_i \sum_{j \neq i} (x_j \times x_i g(|x_j - x_i|) - x_j \times x_i g(|x_j - x_i|)) \\ &= 0. \end{aligned}$$



#2

Show that the system of equations

$$\dot{x} = A \sin(z) + C \cos(y)$$

$$\dot{y} = B \sin(x) + A \cos(z)$$

$$\dot{z} = C \sin(y) + B \cos(x)$$

is volume preserving. Show that when  $A=0$  it has an invariant,  $\Psi(x, y, z) = B \cos(x) + C \sin(y)$ . Discuss the phase portrait in this case.

Solution:

Letting  $F = (A \sin(z) + C \cos(y), B \sin(x) + A \cos(z), C \sin(y) + B \cos(x))$  it follows that

$$\nabla \cdot F = 0,$$

and thus  $F$  is volume preserving. Now if  $A=0$  we have that

$$\frac{d\Psi}{dt} = -B \sin(x) \dot{x} + C \cos(y) \dot{y},$$

$$= -B \sin(x) (C \cos(y)) + C \cos(y) B \sin(x),$$
$$= 0.$$

Consequently, if  $A=0$  it follows that

$$\dot{\Psi} = 0$$

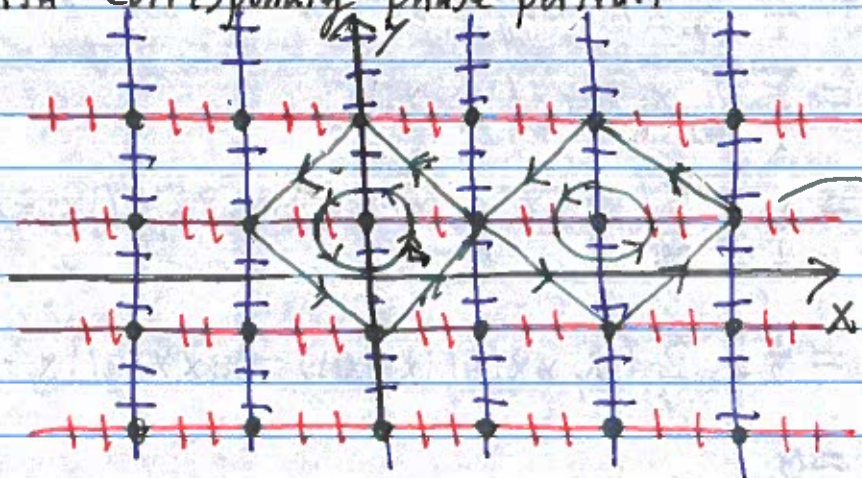
$$\Rightarrow \Psi = \Psi_0 + z_0.$$

The  $x$ - $y$  dynamics are given by:

$$\dot{x} = C \cos(y)$$

$$\dot{y} = B \sin(x)$$

With corresponding phase portrait



and so on...