MST 750
Homework #8

Due Date: April 8, 2022

1. pg. 369, #1
2. pg. 369, #2. (By discuss the phase portrait he means sketch every possible phase portrait).

Homework #8

#1.

Let $x_i \in \mathbb{R}^3$ and consider the system

$$m_i \ddot{x}_i = \sum_{j \neq i} f(x_j - f_i)$$

Where $f: \mathbb{R}^3 \to \mathbb{R}^3$ is the force.

(a) Show that total momentum is conserved if f is odd.

(b) Show that angular momentum, $L = \sum_i m_i \dot{x}_i \times x_i$ is an invariant if $f(x) = x\, g(|x|)$.

Solution:

(a)
$$\frac{dP}{dt} = \sum_{i=1}^{N} m_i \ddot{x}_i = \sum_{i=1}^{N} \sum_{j \neq i}^{N} f(x_j - x_i)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} f(x_j - x_i) + f(x_i - x_j)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \left(f(x_j - x_i) - f(x_j - x_i) \right)$$

$$= 0.$$

(b)
$$\frac{dL}{dt} = \sum_i m_i \ddot{x}_i \times x_i + \sum_i m_i \dot{x}_i \times \dot{x}_i$$

$$= \sum_i \sum_{j \neq i} (x_j - x_i) g(|x_j - x_i|) \times x_i$$

$$= \sum_i \sum_{j \neq i} x_j \times x_i \, g(|x_j - x_i|)$$

$$= \frac{1}{2} \sum_i \sum_{j \neq i} \Big(x_j \times x_i \, g(|x_j - x_i|) + x_i \times x_j \, g(|x_i - x_j|) \Big)$$

$$= \frac{1}{2} \sum_i \sum_{j \neq i} \left(x_j \times x_i \, g(|x_j - x_i|) - x_j \times x_i \, g(|x_j - x_i|) \right)$$

$$= 0. \qquad \blacksquare$$

#2

Show that the system of equations

$$\dot{x} = A\sin(z) + C\cos(y)$$
$$\dot{y} = B\sin(x) + A\cos(z)$$
$$\dot{z} = C\sin(y) + B\cos(x)$$

is volume preserving. Show that when $A=0$ it has an invariant, $\Psi(x,y,z) = B\cos(x) + C\sin(y)$. Discuss the phase portrait in this case.

Solution:

Letting $F = (A\sin(z) + C\cos(y), B\sin(x) + A\cos(z), C\sin(y) + B\cos(x))$ it follows that

$$\nabla \cdot F = 0,$$

and thus F is volume preserving. Now if $A=0$ we have that

$$\frac{d\Psi}{dt} = -B\sin(x)\dot{x} + C\cos(y)\dot{y},$$
$$= -B\sin(x)C\cos(y) + C\cos(y)B\sin(x),$$
$$= 0.$$

Consequently, if $A=0$ it follows that

$$\dot{z} = \Psi_0$$
$$\Rightarrow z = \Psi_0 t + z_0.$$

The x-y dynamics are given by:

$$\dot{x} = C\cos(y)$$
$$\dot{y} = B\sin(x)$$

with corresponding phase portrait