MST 750
Homework #9

Due Date: April 18, 2022

1. pg. 369, #4
2. pg. 369, #5
3. pg. 369, #7

<u>Homework #9</u>

<u>#2.</u>

Verify the standard Poisson bracket and generalized Poisson bracket for the rigid body.

<u>Solution</u>:

I will use summation notation and subscript notation for partial derivatives. For the standard Poisson bracket we have:

$$\{F,G\} = F_{q_i}G_{p_i} - F_{p_i}G_{q_i}$$
$$\{G,H\} = G_{q_i}H_{p_i} - G_{p_i}H_{q_i}$$
$$\{H,F\} = H_{q_i}F_{p_i} - H_{p_i}F_{q_i}$$

Therefore,

$$\{H,\{F,G\}\} = H_{q_k}(F_{q_i p_k}G_{p_i} + F_{q_i}G_{p_i p_k} - F_{p_i p_k}G_{q_i} - F_{p_i}G_{q_i p_k}) - H_{p_k}(F_{q_i q_k}G_{p_i} + F_{q_i}G_{p_i q_k} - F_{q_k p_i}G_{q_i} - F_{p_i}G_{q_i q_k}),$$

$$\{F,\{G,H\}\} = F_{q_k}(G_{q_i p_k}H_{p_i} + G_{q_i}H_{p_i p_k} - G_{p_i p_k}H_{q_i} - G_{p_i}H_{q_i p_k}) - F_{p_k}(G_{q_i q_k}H_{p_i} + G_{q_i}H_{p_i q_k} - G_{q_k p_i}H_{q_i} - G_{p_i}H_{q_i q_k}),$$

$$\{G,\{H,F\}\} = G_{q_k}(H_{q_i p_k}F_{p_i} + H_{q_i}F_{q_k p_i} - H_{p_i p_k}F_{q_i} - H_{p_i}F_{q_i p_k}) - G_{p_k}(H_{q_i q_k}F_{p_i} + H_{q_i}F_{p_i q_k} - H_{q_k p_i}F_{q_i} - H_{p_i}F_{q_i q_k}).$$

Therefore,

$$\{H,\{F,G\}\} + \{F,\{G,H\}\} + \{G,\{H,F\}\}$$
$$= H_{q_k}(\cancel{F_{q_i p_k}G_{p_i}} + F_{q_i}\cancel{G_{p_i p_k}} - \cancel{F_{p_i p_k}}G_{q_i} - \cancel{F_{p_i}G_{q_i p_k}} - \cancel{F_{q_i}G_{p_i p_k}} - \cancel{F_{p_i}G_{q_i p_k}}$$
$$+ \cancel{G_{q_i}F_{q_i p_k}} - \cancel{G_{p_i}F_{p_k q_i}})$$
$$+ H_{p_k}(\quad) + F_{q_k}(\quad) + F_{p_k}(\quad) + G_{q_k}(\quad) + G_{p_k}(\quad).$$
$$= 0$$

For the rigid body the bracket is defined by:

$$\{F,G\} = [F_{L_1}, F_{L_2}, F_{L_3}]\begin{bmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{bmatrix}\begin{bmatrix} G_{L_1} \\ G_{L_2} \\ G_{L_3} \end{bmatrix}$$

$$= [F_{L_1}, F_{L_2}, F_{L_3}]\begin{bmatrix} -L_3 G_{L_2} + L_2 G_{L_3} \\ L_3 G_{L_1} - L_1 G_{L_3} \\ -L_2 G_{L_1} + L_1 G_{L_1} \end{bmatrix}$$

$$= F_{L_1}(L_2 G_{L_3} - L_3 G_{L_2}) + F_{L_2}(L_3 G_{L_1} - L_1 G_{L_3})$$

[illegible]

Therefore,

$$\{F,G\} = \nabla_L F\cdot(\vec{L}\times\nabla_L G),\ \{G,H\} = \nabla_L G\cdot(\vec{L}\times\nabla_L H),\ \{H,F\} = \nabla_L H\cdot(\vec{L}\times$$

and thus

$$\{H,\{F,G\}\}+\{F,\{G,H\}\}+\{G,\{H,F\}\}$$

$$=\nabla_L H\cdot(\vec{L}\times\nabla_L(\nabla_L F\cdot(\vec{L}\times\nabla_L G)))$$
$$+\nabla_L F\cdot(\vec{L}\times\nabla_L(\nabla_L G\cdot(\vec{L}\times\nabla_L H)))$$
$$+\nabla_L G\cdot(\vec{L}\times\nabla_L(\nabla_L H\cdot(\vec{L}\times\nabla_L F)))$$

$$=\nabla_L H\cdot(\vec{L}\times(\nabla_L^2 F(\vec{L}\times\nabla_L G)+\nabla_L(\vec{L}\times\nabla_L G)\nabla_L F))$$
$$+\nabla_L F\cdot(\vec{L}\times(\nabla_L^2 G(\vec{L}\times\nabla_L H)+\nabla_L(\vec{L}\times\nabla_L H)\nabla_L G))$$
$$+\nabla_L G\cdot(\vec{L}\times(\nabla_L^2 H\cdot(\vec{L}\times\nabla_L F)+\nabla_L(\vec{L}\times\nabla_L F)\nabla_L H))$$

$$=\nabla_L H\cdot(\vec{L}\times(\nabla_L^2 F(\vec{L}\times\nabla_L G)+(\nabla_L\vec{L}\times\nabla_L G-\vec{L}\times\nabla_L^2 G)\nabla_L F))$$
$$+\nabla_L F\cdot(\vec{L}\times(\nabla_L^2 G(\vec{L}\times\nabla_L H)+(\nabla_L\vec{L}\times\nabla_L H-L\times\nabla_L^2 H)\nabla_L G))$$
$$+\nabla_L G\cdot(\vec{L}\times(\nabla_L^2 H(\vec{L}\times\nabla_L F)+(\nabla_L\vec{L}\times\nabla_L F-L\times\nabla_L^2 F)\nabla_L H))$$

$$=\nabla_L H\cdot(\vec{L}\times\nabla_L^2 F(\vec{L}\times\nabla_L G))-\nabla_L G\cdot(L\times\nabla_L^2 F(\nabla_L H))$$
$$+\nabla_L F\cdot(\vec{L}\times\nabla_L^2 G(\vec{L}\times\nabla_L H))-\nabla_L H\cdot(L\times\nabla_L^2 G(\nabla_L F))$$
$$+\nabla_L G\cdot(\vec{L}\times\nabla_L^2 H(\vec{L}\times\nabla_L F))-\nabla_L F\cdot(L\times\nabla_L^2 H)(\nabla_L G))$$
$$+\nabla_L H\cdot((\nabla_L\vec{L}\times\nabla_L G)\nabla_L F)+\nabla_L F\cdot((\nabla_L\vec{L}\times\nabla_L H)\nabla_L G))+\nabla_L G((\nabla_L\vec{L}\times\nabla_L F)\nabla_L H$$

$$=0 \leftarrow$$ I guess.

#3.

Poisson systems often have a special invariant, called a Casimir, a non-constant function $C\in C^1(M,\mathbb{R})$ such that $\{C,F\}=$ for any $F\in C^1(M,\mathbb{R})$.

(a) Show that if C is a Casimir for the bracket defined by

$$\{F,G\}=\nabla F^T J\,\nabla G$$

then J is singular.

(b) Show if $\dim(M)$ is odd then since the Poisson bracket is antisymmetric, it must have at least one local Casimir

(c). Show that the canonical braket does not have a Casimir

(d) The rigid body bracket has one Casimir. Find it.
(e). Describe the dynamics of Euler's equations.

Solution:

(a) Suppose C is a Casimir. Then for all F it follows that
$$\nabla F^T J \nabla C = 0,$$
$$\Rightarrow F_{,i} J_{ij} C_{,j} = 0.$$
Selecting F so that $F_{,i} = \delta_{ik}$ yields
$$J_{kj} C_{,j} = 0,$$
that is ∇C is orthogonal to all of the columns of J. This is only possible if J is singular, i.e., the columns of J are linearly dependent.

(b). If J is odd then it is singular since
$$\det(J) = \det(J^T) = \det(-J) = (-1)^n \det(J)$$
$$\Rightarrow \det(J) = 0.$$
To construct the casimir we must choose C so that ∇C is everywhere orthogonal to the columns

(c) The canonical bracket cannot have a bracket because it is nonsingular.

(d) For the rigid body:
$$J = \begin{bmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{bmatrix}$$
Consequently, the equations for a Casimir are:
$$L_3 \frac{\partial C}{\partial L_2} - L_2 \frac{\partial C}{\partial L_3} = 0$$
$$-L_3 \frac{\partial C}{\partial L_1} + L_1 \frac{\partial C}{\partial L_3} = 0$$
$$L_2 \frac{\partial C}{\partial L_1} - L_1 \frac{\partial C}{\partial L_2} = 0$$
$$\Rightarrow \frac{1}{L_3}\frac{\partial C}{\partial L_3} = \frac{1}{L_2}\frac{\partial C}{\partial L_2}, \quad \frac{1}{L_1}\frac{\partial C}{\partial L_1} = \frac{1}{L_3}\frac{\partial C}{\partial L_3}, \quad \frac{1}{L_2}\frac{\partial C}{\partial L_2} = \frac{1}{L_1}\frac{\partial C}{\partial L_1}$$
$$\Rightarrow C = L_1^2 + L_2^2 + L_3^2 \text{ is a Casimir.}$$