Read:
Sections 1.1–1.3, 1.5.

Lecture #1: One Dimensional Dynamics

Autonomous Differential Equation

$\frac{dx}{dt} = \dot{x} = f(x)$, $x: \mathbb{R} \to \mathbb{R}$, f is smooth.

$\frac{dx}{dt}$: velocity $f(x)$: function of position.

$\underbrace{x(t_0) = x_0}_{\text{initial position}}$

Example:

$\dot{x} = 3x$
$x(0) = 4$
$\to x(t) = 4e^{3t}$

Check:

$\dot{x} = 12e^{3t} = 3\cdot(4e^{3t}) = 3\cdot x$ ✓
$x(0) = 4e^{0} = 4$ ✓

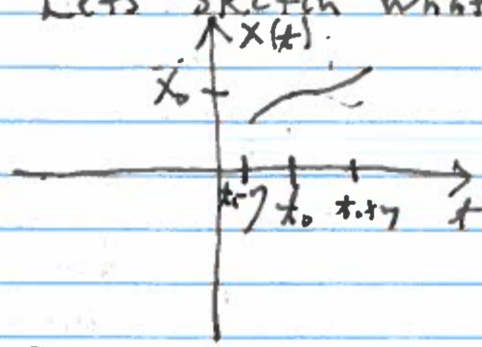
Q: Is this the only solution??

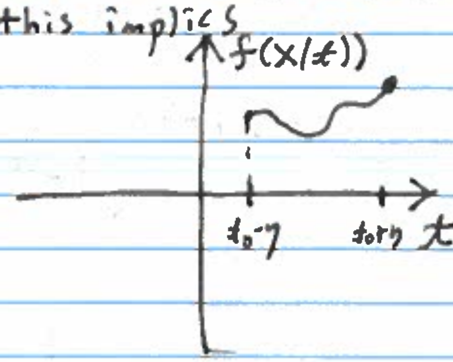
Uniqueness:

Suppose $x(t)$ solves:

$\dot{x} = f(x)$,
$x(t_0) = x_0$,

on some interval of time $[t_0 - \eta, t_0 + \eta]$ and $f(x_0) \neq 0$,
Lets sketch what this implies





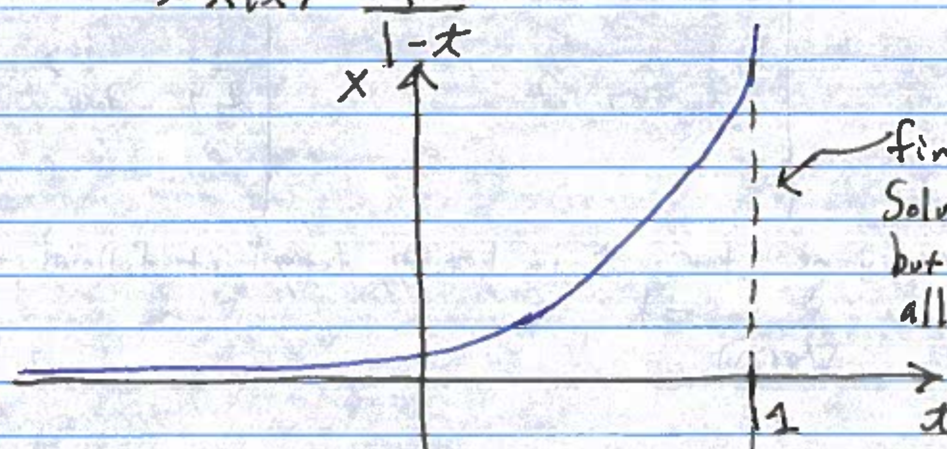
Therefore, for t in $[t_0 - \eta, t_0 + \eta]$ it follows that

$$\frac{\dot{x}}{f(x(t))} = 1.$$

Therefore,
$$\int_{t_0}^{t} \frac{1}{f(x(t))}\frac{dx}{dt}\,dt = \int_{t_0}^{t} dt$$
$$\Rightarrow \int_{x_0}^{x} \frac{1}{f(x)}\,dx = t - t_0$$
Letting $F(x) = \int_{x_0}^{x} \frac{1}{f(x)}\,dx$ it follows that F is monotone in x since f is of one sign, i.e. $F'(x) = 1/f(x)$. Therefore, F is invertible and thus
$$x = F^{-1}(t - t_0),$$
is the unique solution.

■

Example:
$$\dot{x} = x^2$$
$$x(0) = 1$$
$$\Rightarrow \int_0^t \frac{\dot{x}}{x^2}\,dt = \int_0^t \frac{1}{x(t)^2}\frac{dx}{dt}\,dt = \int_0^t dt$$
$$\Rightarrow \int_1^x \frac{1}{x^2}\,dx = t$$
$$-\frac{1}{x}\Big|_1^x = t$$
$$\Rightarrow 1 - \frac{1}{x} = t$$
$$\Rightarrow 1 - t = \frac{1}{x}$$
$$\Rightarrow x(t) = \frac{1}{1-t}$$

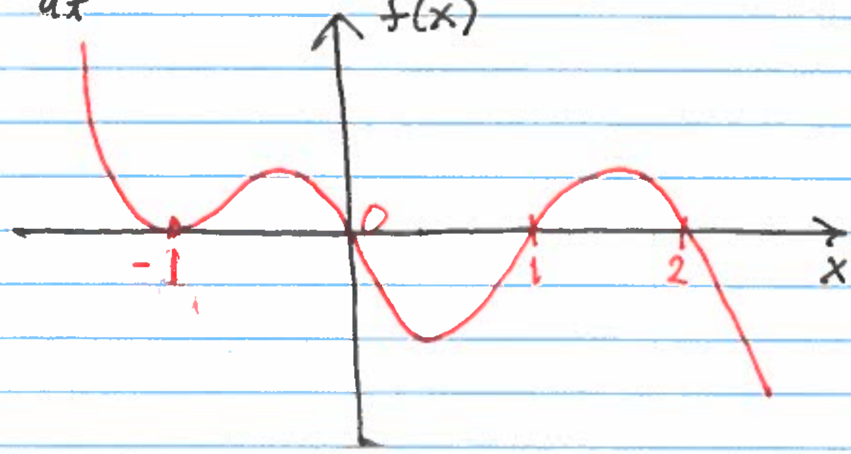


finite time blow up

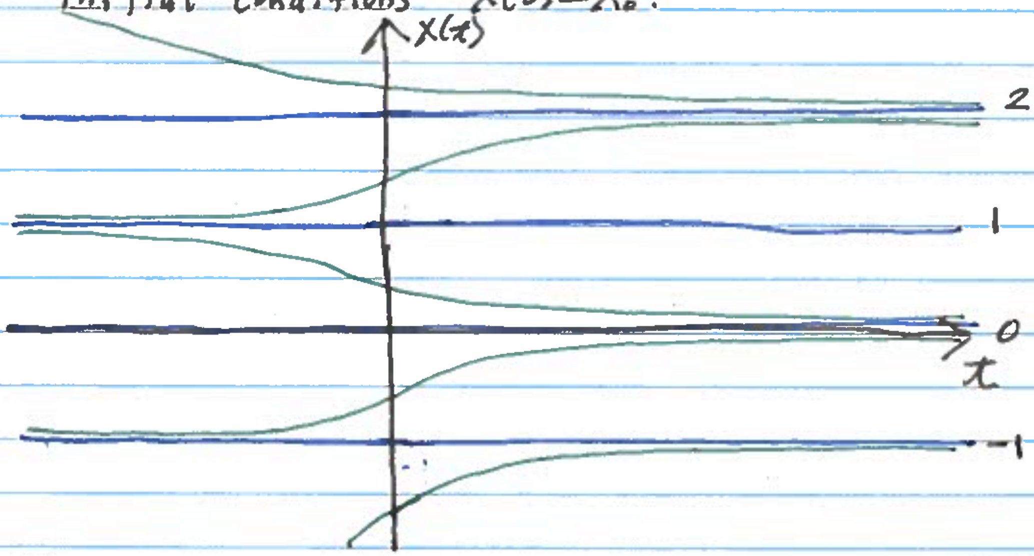
Solutions might be unique but not exist for all time!

Geometric Approach:

$$\frac{dx}{dt} = f(x)$$



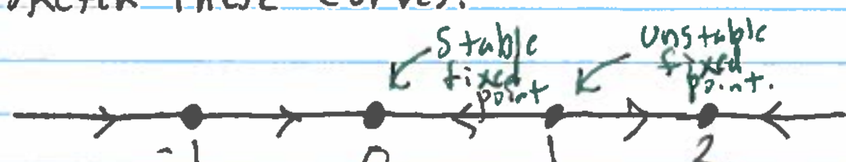
- $x(t) = -1, 0, 1, 2$ are exact solutions.
- How can we sketch solution curves for different initial conditions $x(0) = x_0$?



* Note changes in concavity occur when $\ddot{x}(t) = 0$.

$$\ddot{x} = \frac{d}{dt} f(x) = f'(x)\cdot\dot{x} = f'(x)\, f(x).$$

All we really needed is the "topology" of f to sketch these curves.



This system is "topologically" equivalent to: