Lecture 10: Contraction Mapping Theorem and Existence and Uniqueness

Contraction Maps

Let $T:(X,\|\cdot\|)\to(X,\|\cdot\|)$ be a map on a Banach space $(X,\|\cdot\|)$. The map is a contraction if there exists $c<1$ such that for all $x,y\in X$:

$$\|Tx-Ty\|\leq c\|x-y\|.$$

In this case T has a unique fixed point x^* such that

$$T(x^*)=x^*.$$

proof:

Consider the sequence

$$x_{n+1}=Tx_n.$$

It follows that for $m>n\geq 1$

$$\begin{aligned}
\|x_m-x_n\| &= \|x_m-x_{m-1}+x_{m-1}+\dots+x_{n+1}-x_n\| \\
&\leq \|x_m-x_{m-1}\|+\dots+\|x_{n+1}-x_n\| \\
&= \|Tx_{m-1}-Tx_{m-2}\|+\dots+\|x_{n+1}-x_n\| \\
&\leq c\|x_{m-1}-x_{m-2}\|+\dots+\|x_{n+1}-x_n\| \\
&\leq c^{m-n}\|x_{n+1}-x_n\|+c^{m-n-1}\|x_{n+1}-x_n\|+\dots+\|x_{n+1}-x_n\| \\
&= (c^{m-n}+c^{m-n-1}+\dots+c+1)\|x_{n+1}-x_n\| \\
&\leq (c^{m-n}+c^{m-n-1}+\dots+c+1)c^n\|x_1-x_0\| \\
&\leq \frac{c^n}{1-c}\|x_1-x_0\|
\end{aligned}$$

Therefore, $\lim_{n\to\infty}\|x_m-x_n\|=0$ and thus x_m is Cauchy. Consequently, there exists x^* such that $\lim_{n\to\infty}x_n=x^*$ and since T is continuous it follows that

$$T(x^*)=T(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}T(x_n)=\lim_{n\to\infty}x_{n-1}=x^*.$$

To prove uniqueness suppose $T(x^*)=x^*$ and $T(y^*)=y^*$. Therefore

$$\|Tx^*-Ty^*\|\leq c\|x^*-y^*\|$$

$$\Rightarrow \|x^*-y^*\|\leq\|x^*-y^*\|c$$

$$\Rightarrow c\geq 1$$

A contradiction.

Lipschitz Functions

Suppose $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are normed linear spaces.

- $f: \mathbb{X} \to \mathbb{Y}$ is Lipshitz continuous if for all $x_1, x_2 \in \mathbb{X}$
$$\|f(x_1) - f(x_2)\|_{\mathbb{Y}} \leq K\|x_1 - x_2\|_{\mathbb{X}}$$
for some $K > 0$. The smallest such K is called the Lipshitz constant.
- $f: \mathbb{X} \to \mathbb{Y}$ is locally Lipshitz if for every $x_1 \in \mathbb{X}$ there exists a ball, $B_r = \{x \in \mathbb{X} : \|x - x_1\| < r\}$, such that f is Lipschitz on B_r.

Example:

Let $T: \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. Therefore for all $x_1, x_2 \in \mathbb{X}$
$$\|Tx_1 - Tx_2\|_{\mathbb{Y}} = \|T(x_1 - x_2)\|_{\mathbb{Y}} \leq \|T\|_{\mathbb{X}} \|x_1 - x_2\|.$$
Therefore, T is Lipshitz and the inequality is strict and thus the Lipshitz constant $\|T\|_{\mathbb{X}}$.

Theorem - If $f: \mathbb{X} \to \mathbb{Y}$ is locally Lipshitz and $A \subset \mathbb{X}$ is compact then f is Lipschitz on A.

Theorem - Suppose $A \subset \mathbb{R}^n$ is compact and convex and $f \in C^1(A, \mathbb{R}^n)$. Then f is Lipschitz with constant $K = \max_{x \in A} \|Df\|$.

proof:

Let $x, y \in A$. Since A is convex
$$\bar{x}(s) = sx + (1-s)y \in A$$
for $0 \leq s \leq 1$. Therefore,
$$f(y) - f(x) = \int_0^1 \frac{d}{ds} f(\bar{x}(s))\,ds = \int_0^1 Df(\bar{x}(s))(x-y)\,ds$$
$$\Rightarrow \|f(y) - f(x)\| \leq \|Df(s)\|_\infty \|x - y\|$$

Theorem - Suppose for $x_0 \in \mathbb{R}^n$ there is a $b > 0$ such that $f: B_r(x_0) \to \mathbb{R}^n$ is Lipschitz with constant K. Then there exists $a > 0$ such that

$$\dot{x} = f(x)$$
$$x(t_0) = x_0$$

has a unique solution for $t \in J = [t_0 - a, t_0 + a]$

~~Proof~~:

Let $V = C^0(J, B_b(x_0))$ and define $T: V \to V$ by

$$T(x(t)) = x_0 + \int_{t_0}^{t} f(x(t))\,dt.$$

We first prove that $T(x(t)) \in V$.

$$\|T(x)(t) - x_0\| \leq \int_{t_0}^{t} |f(x(t))|\,dt \leq M|t - t_0| \leq Ma.$$

Pick a so that $Ma < b$. We also have

$$\|Tx(t) - Ty(t)\| \leq \int_{t_0}^{t} \|f(x(t)) - f(y(t))\|\,dt$$
$$\leq K \int_{t_0}^{t} \|x(\tau) - y(\tau)\|\,d\tau$$
$$\leq Ka\|x - y\|.$$

If we pick $aK < 1$ we have a contraction.

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