Lecture 2: Two Dimensional Dynamics

$\dot{\vec{x}} = F(\vec{x})$, $F: \mathbb{R}^n \to \mathbb{R}^n$ is smooth.
$\vec{x}(0) = \vec{x}_0$

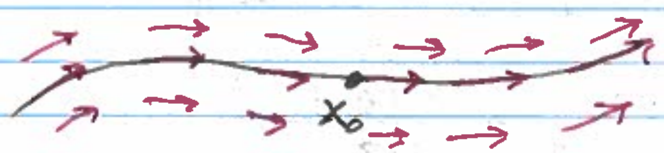
Tangent vector

Vector Field

Theorem:
- Solutions exist
- Solutions are unique
- ☆ Solutions depend continuously on initial data.

$\Downarrow$

We can view differential equations as mappings of initial conditions to curves.



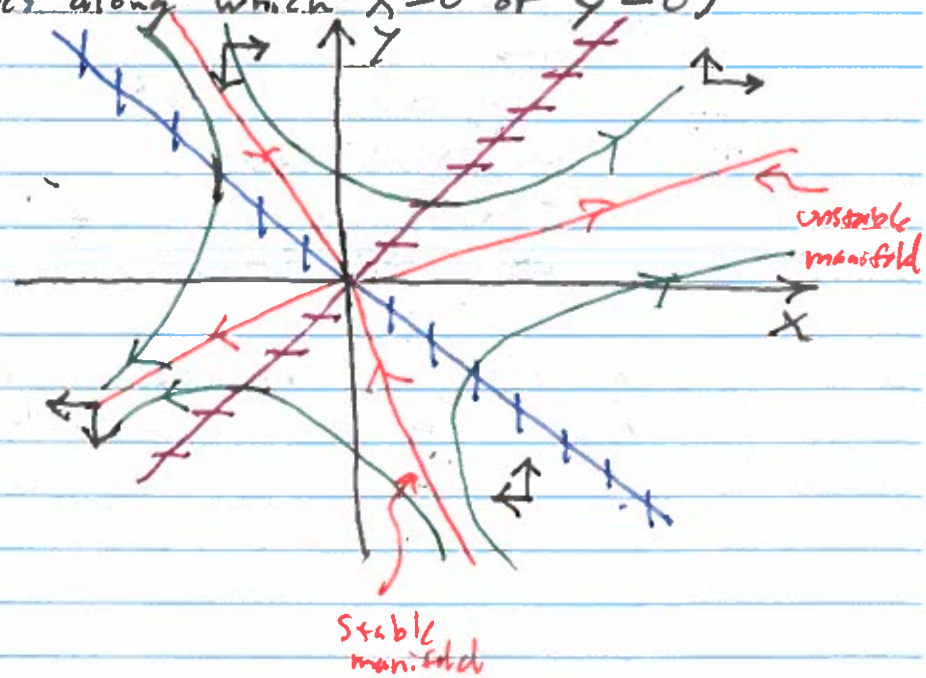
Example:

$\dot{x} = \exp(x+y)(x+y)$
$\dot{y} = \exp(x+y)(x-y)$

Nullclines (curves along which $\dot{x}=0$ or $\dot{y}=0$)

N1 ($\dot{x}=0$):
$y = -x$

N2 ($\dot{y}=0$):
$y = x$



If we want to solve for $(x(t), y(t))$ one idea is to think of y as a function of x:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x-y}{y+x}$$

Let $z = y/x$. Then

$$\frac{dz}{dx} = \frac{x\frac{dy}{dx} - y}{x^2}$$

$$= \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

$$= \frac{1}{x}\left(\frac{x-y}{y+x}\right) - \frac{y}{x^2}$$

$$= \frac{1}{x}\left(\frac{1-z}{1+z} - z\right)$$

$$= \frac{1}{x}\left(\frac{1-z-z-z^2}{1+z}\right)$$

$$= \frac{1}{x}\frac{(1-2z-z^2)}{1+z}$$

Solving this O.D.E. yields

$$(y+x)^2 - 2x^2 = E$$

constant.

hyperbolas for different energy levels.

$E(0,0) = 0$, so curves passing through the origin satisfy

$$0 = (y+x)^2 - 2x^2$$

$$\Rightarrow 2x^2 = (y+x)^2$$

$$\Rightarrow \pm\sqrt{2}x = y+x$$

$$\Rightarrow y = (\pm\sqrt{2}-1)x$$

Regardless, we can think of solution curves as contours of the function $E(x,y)$.

Example:

$$\dot{x} = y + x\cos x$$
$$\dot{y} = -\sin(x)$$

N1:

$$\dot{x} = 0 \Rightarrow y = -x\cos(x)$$

N2:

$$\dot{y} = 0 \Rightarrow x = n\pi$$

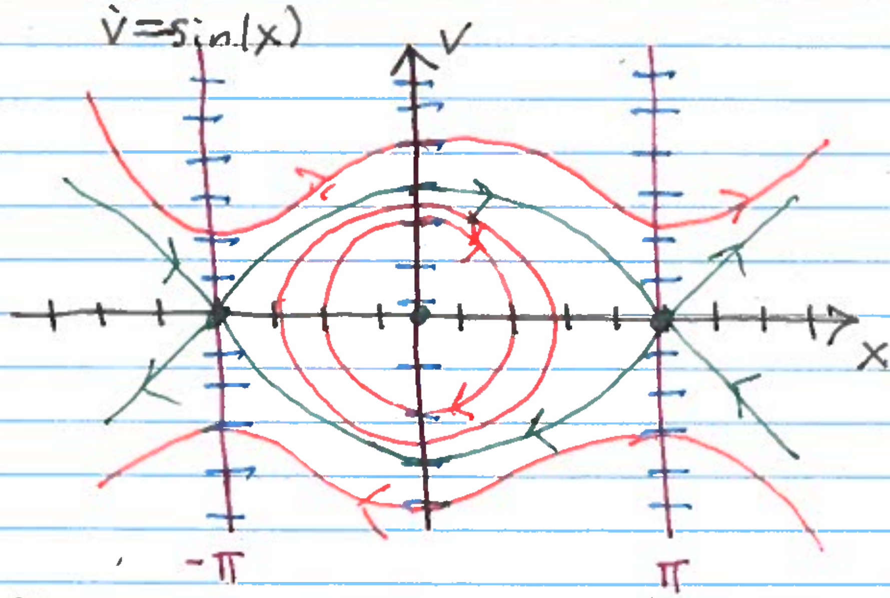
*Refer to plot plot in Mathematica.

Example:

$$\ddot{x} = \sin(x)$$

Let $v = \dot{x}$ we have:

$$\dot{x} = v$$
$$\dot{v} = \sin(x)$$



Multiplying both sides by $\dot{x}$:

$$\dot{x}\ddot{x} = \dot{x}\sin(x)$$
$$\frac{d}{dt}\frac{1}{2}\dot{x}^2 = \frac{d}{dt}(+\cos(x))$$
$$\Rightarrow \frac{1}{2}\dot{x}^2 - \cos(x) = E \text{ is a constant.}$$

$E(0,\pi) = 1$, therefore the heteroclinic orbit satisfies

$$\frac{1}{2}\dot{x}^2 - \cos(x) = 1$$

Therefore, the heteroclinic orbit connecting $-\pi$ to π is given by:

$$v = \sqrt{2(1+\cos(x))}$$