Lecture 7: Nonautonomous Linear Systems

Example:

$$\dot{\vec{x}} = A\vec{x} + f(t), \quad \vec{x}(t_0) = \vec{x}(0)$$

$$\Rightarrow \dot{\vec{x}} - A\vec{x} = f(t)$$

$$\Rightarrow (e^{-At}\dot{\vec{x}} - e^{-At}\vec{x}) = e^{-At} f(t)$$

$$\Rightarrow \frac{d}{dt}(e^{-At}\vec{x}) = e^{-At} f(t)$$

$$\Rightarrow \int_{\vec{x}_0}^{e^{-At}\vec{x}} d(e^{-At}\vec{x}) = \int_{t_0}^{t} e^{-As} f(s)\,ds$$

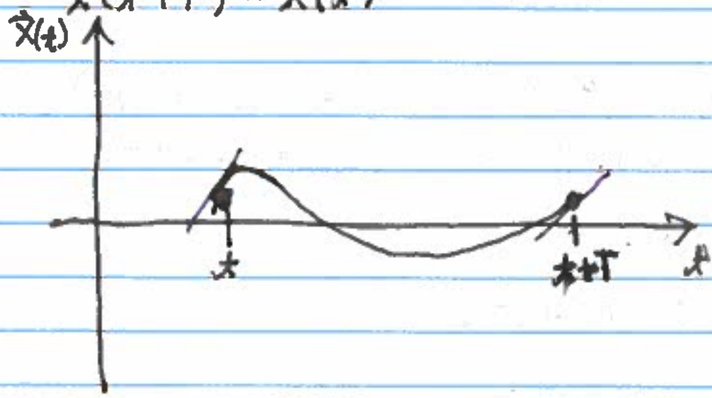
$$\Rightarrow e^{-At}\vec{x}(t) - \vec{x}_0 = \int_{t_0}^{t} e^{-A(s)} f(s)\,ds$$

$$\Rightarrow \vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_{t_0}^{t} e^{-A(s)} f(s)\,ds$$

Periodic solutions satisfy:

$$\vec{x}(t+T) = \vec{x}(t) \Rightarrow f(t+T) = f(t).$$

$$\dot{\vec{x}}(t+T) = \dot{\vec{x}}(t)$$



For simplicity, assume $t_0 = 0$, $T = 1$. (Can introduce change of variables $\tau = (t - t_0)/T$.)

$$\Rightarrow \vec{x}(T) = e^{AT}\vec{x}_0 + e^{AT}\int_0^T e^{-As} f(s)\,ds = \vec{x}_0$$

$$\Rightarrow \boxed{\vec{x}_0(I - e^{AT}) + \int_0^T e^{-As} f(s)\,ds = 0} \Rightarrow \boxed{\vec{x}_0 = (I - e^{AT})^{-1}\int_0^T e^{-As} f(s)\,ds}$$

Define a map $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(\vec{x}_0) = e^{AT}\vec{x}_0 + e^{AT}\int_0^T e^{-As} f(s)\,ds.$$

Periodic orbits satisfy:

$$\boxed{f(\vec{x}_0) = \vec{x}_0} \longrightarrow \text{Equation for a fixed point.}$$

If there is a solution $x^*(t)$ that is periodic let

$$y(t) = x - x^*$$
$$\Rightarrow x = y + x^*$$
$$\Rightarrow \dot{y} = \dot{x} - \dot{x}^*$$
$$\Rightarrow \dot{y} = Ax + f(t) - Ax^* - f(t)$$
$$\Rightarrow \boxed{\dot{y} = Ay.}$$

Therefore, if eigenvalues of A satisfy $\mathrm{Re}(\lambda_i) < 0$ then all solutions satisfy

$$\lim_{t\to\infty} x(t) = x^*(t).$$

Example:

$$\dot{\vec{x}} = A(t)\vec{x}$$
$$\vec{x}(0) = \vec{x}_0$$

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2(t) & 1 - \alpha\cos(t)\sin(t) \\ -1 - \alpha\cos(t)\sin(t) & -1 + \alpha\sin^2(t) \end{pmatrix}$$

$$\Rightarrow \dot{\vec{x}} - A(t)\vec{x} = 0$$
$$\Rightarrow e^{-A(t)}\dot{\vec{x}} - e^{-A(t)}A(t)\vec{x} = 0$$

However,

$$\frac{d}{dt}\left(e^{-\int A(t)}\vec{x}\right) = e^{-\int A(t)}\dot{\vec{x}} - A(t)e^{-\int A(t)}\vec{x}$$
$$\neq e^{-\int A(t)}\dot{\vec{x}} - e^{-\int A(t)}A(t)\vec{x}!$$

$\Rightarrow$ stability is not trivial!

Indeed, the eigenvalues are given by

$$\lambda = \frac{1}{2}\left(\alpha - 2 \pm \sqrt{\alpha^2 - 4}\right).$$

If $\alpha < 2 \Rightarrow Re(\lambda) < 0$, however

$$x_1(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} \exp((\alpha - 1)t),$$

$$x_2(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \exp(-t),$$

and thus the system is unstable if $\alpha > 1$.