MTH 357/657
Classwork #7

April 25, 2023

1. If X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find the mean and standard deviation for $Z = 1/2(Y - 3X)$.

2. When current I flows through resistance R, the power generated is given by $W = I^2R$. Suppose I has a uniform distribution over the interval $(0, 1)$ and R has a density function given by
$$f(r) = \begin{cases} 2r & 0 \le r \le 1 \\ 0 & \text{elsewhere} \end{cases},$$
where $r > 0$. Assuming R and I are independent find the following:
   (a) The mean and standard deviation of W,
   (b) The probability density function for W.

3. Let X and Y be independent and uniformly distributed over the interval $(0, 1)$. Find the probability density function for each of the following random variables:
   (a) $Z_1 = X/Y$,
   (b) $Z_2 = -\ln(XY)$,
   (c) $Z_3 = XY$.

4. A random variable X has a beta distribution of the second kind, if, for $\alpha > 0$ and $\beta > 0$, its density is
$$f(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\beta(\alpha,\beta)(1+x)^{\alpha+\beta}}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}.$$
Find the probability density of the random variable
$$Y = \frac{1}{1+X}.$$

#1

If X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find the mean and standard deviation for $Z = \frac{1}{2}(Y - 3X)$.

Solution:

(a) $\mathbb{E}[Z] = \mathbb{E}[\frac{1}{2}(Y - 3X)]$

$$= \frac{1}{2}\mathbb{E}[Y] - \frac{3}{2}\mathbb{E}[X]$$

$$= \frac{1}{2}\mu - \frac{3}{2}\mu$$

$$= -\mu.$$

(b). $\sigma_Z^2 = \mathrm{Cov}(Z, Z)$

$$= \mathrm{Cov}(\tfrac{1}{2}(Y-3X), \tfrac{1}{2}(Y-3X))$$

$$= \tfrac{1}{4}\mathrm{Cov}(Y, Y) + \tfrac{9}{4}\mathrm{Cov}(X, X)$$

$$= {}^{10}/_{4}\,\sigma^2$$

$$\Rightarrow \sigma_Z = \sqrt{10}/2\,\sigma.$$

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#2

When current I flows through resistance R, the power generated is given by $W = I^2 R$. Suppose I has a uniform distribution over the interval $(0,1)$ and R has the density function given by

$$f(r) = \begin{cases} 2r, & 0 \le r \le 1 \\ 0, & \text{elsewhere} \end{cases},$$

Where $r > 0$. Assuming R and I are independent find the following:

(a) The mean and standard deviation of W.

(b) The probability density for W.

Solution:

(a) $\mathbb{E}[W] = \mathbb{E}[I^2 R]$

$$= \mathbb{E}[I^2]\,\mathbb{E}[R]$$
$$= \left(\int_0^1 i^2\,di\right)\left(\int_0^1 2r^2\,dr\right)$$
$$= \tfrac{1}{3}\cdot\tfrac{2}{3}$$
$$= \tfrac{2}{9}.$$

$\mathbb{E}[W^2] = \mathbb{E}[I^4 R^2]$

$$= \mathbb{E}[I^4]\,\mathbb{E}[R^2]$$
$$= \left(\int_0^1 i^4\,di\right)\left(\int_0^1 2r^3\,dr\right)$$
$$= \tfrac{1}{5}\cdot\tfrac{1}{2}$$
$$= \tfrac{1}{10}.$$

Therefore,

$$\sigma_W^2 = \left(\tfrac{1}{10} - \tfrac{4}{81}\right)^{1/2}.$$

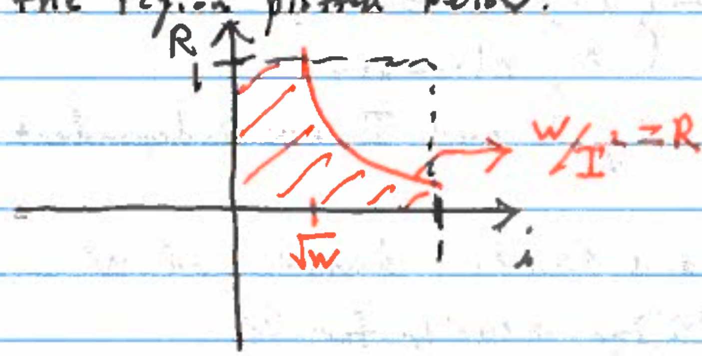
(b). $F(w) = P(W \leq w)$

$$= P(I^2 R \leq w)$$
$$= P(R \leq w/I^2).$$

Now, since I and R are independent the joint density is given by

$$p(i,r) = \begin{cases} 2r, & 0<i<1,\ 0<r<1, \\ 0, & \text{elsewhere} \end{cases}$$

Consequently, if $0 \leq W \leq 1$ we have that the relevant integral is over the region plotted below:



Therefore, if $0 \le w \le 1$, we have that

$$P(R \le w/\text{I}^2) = \int_0^{\sqrt{w}} \int_0^1 2r\,dr\,di + \int_{\sqrt{w}}^1 \int_0^{w/i^2} 2r\,dr\,di$$

$$= \int_0^{\sqrt{w}} di + \int_{\sqrt{w}}^1 w^2/i^4\,di.$$

Consequently,

$$F(w) = \begin{cases} 0, & w \le 0 \\ \sqrt{w} + \int_{\sqrt{w}}^1 w^2/i^4\,di, & 0 \le w \le 1 \\ 1, & w \ge 1 \end{cases}$$

$$= \begin{cases} 0, & w \le 0 \\ \sqrt{w} + w^2\left(-\frac{1}{3}\frac{1}{i^3}\Big|_{\sqrt{w}}^1\right), & 0 \le w \le 1 \\ 1, & w \ge 1 \end{cases}$$

$$= \begin{cases} 0, & w \le 0 \\ \sqrt{w} + w^2\left(\frac{1}{3}\left(\frac{1}{w^{3/2}} - 1\right)\right), & 0 \le w \le 1 \\ 1, & w \ge 1 \end{cases}$$

$$= \begin{cases} 0, & w \le 0 \\ \sqrt{w} + \frac{1}{3}\sqrt{w} - \frac{1}{3}w^2, & 0 \le w \le 1 \\ 1, & w \ge 1 \end{cases}$$

$$\Rightarrow \frac{dF}{dW} = \begin{cases} \frac{2}{3}w^{-1/2} - \frac{2}{3}w, & 0 \le w \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$\blacksquare$

#3.

Let X and Y be independent and uniformly distributed over the interval $(0,1)$. Find the probability density function for each of the following random variables:

(a) $Z_1 = X/Y$

(b) $Z_2 = -\ln(XY)$

(c) $Z_3 = X \cdot Y$.

Solution:

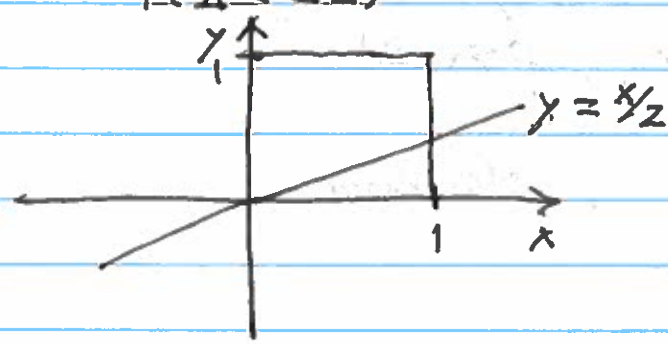
For all of the problems, the joint density is given by

$$p(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

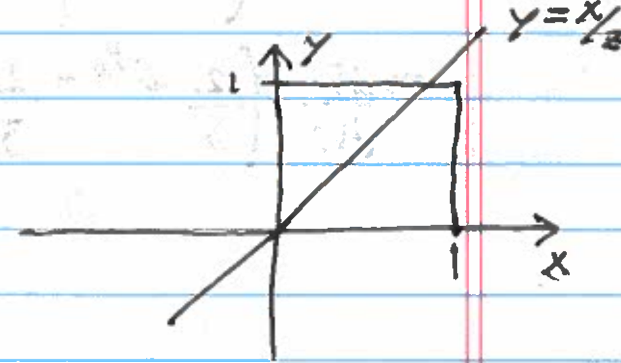
(a) $F(z) = P(Z \le z)$

$= P(X/Y < z)$

$= P(X < Yz)$



Case 1: $z>1$



Case 2: $0<z<1$

$$\Rightarrow P(X<Yz) = \begin{cases} 0 & z<0 \\ \int_0^z \int_{x/z}^1 1\,dy\,dx & 0<z<1 \\ \int_0^1 \int_{x/z}^1 1\,dy\,dx & z>1 \end{cases}$$

$$\Rightarrow F(z) = \begin{cases} 0, & z < 0 \\ \int_0^z (1 - x/z)\,dx & 0 < z < 1 \\ \int_0^1 (1 - x/z)\,dx & z > 1 \end{cases}$$

$$= \begin{cases} 0, & z < 0 \\ z - \tfrac{1}{2} z, & 0 < z < 1 \\ 1 - \frac{1}{2z}, & z > 1 \end{cases}$$

$$= \begin{cases} 0, & z \leq 0 \\ \tfrac{1}{2} z, & 0 \leq z \leq 1 \\ 1 - \frac{1}{2z}, & z \geq 1 \end{cases}$$

Consequently, the density is given by

$$f(z) = \frac{dF}{dz} = \begin{cases} \tfrac{1}{2} & 0 \leq z \leq 1 \\ \frac{1}{2z^2}, & z \geq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

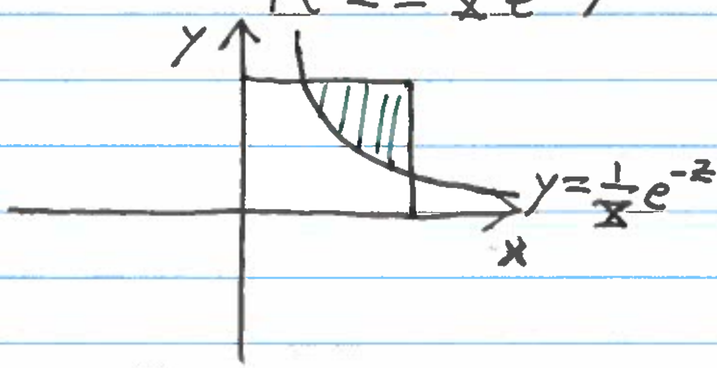
(b) $F(z) = P(Z \leq z)$

$$= P(-\ln(XY) \leq z)$$

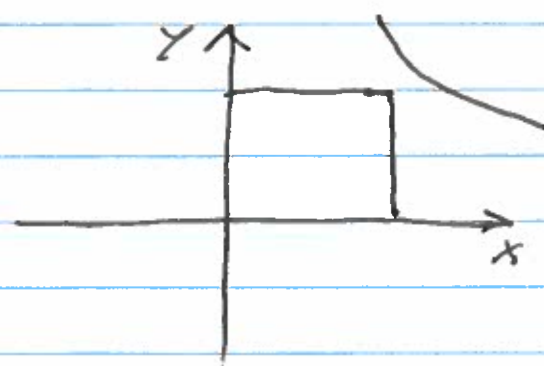
$$= P(\ln(XY) \geq -z)$$

$$= P(XY \geq e^{-z})$$

$$= P(Y \geq \tfrac{1}{X} e^{-z})$$



$z > 0$



$z < 0$

Therefore,

$$P\left(Y \ge \tfrac{1}{X}e^{-z}\right) = \begin{cases} 0, & z<0 \\ \int_{e^{-z}}^{1} \int_{\frac{1}{x}e^{-z}}^{1} 1 \, dy \, dx & z>0 \end{cases}$$

$$= \begin{cases} 0, & z<0 \\ \int_{e^{-z}}^{1} \left(1 - \tfrac{1}{x}e^{-z}\right) dx, & z>0 \end{cases}$$

$$= \begin{cases} 0, & z<0 \\ 1-e^{-z} + \ln(x)e^{-z} \Big|_{e^{-z}}^{1}, & z>0 \end{cases}$$

$$= \begin{cases} 0, & z<0 \\ 1-e^{-z}+ze^{-z}, & z>0. \end{cases}$$

Consequently,

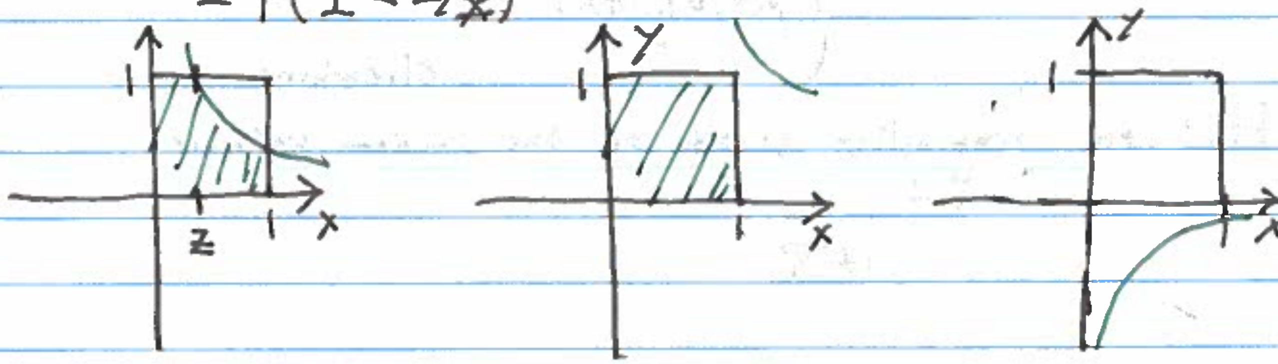
$$f(z) = \frac{dF}{dz} = \begin{cases} e^{-z}-e^{-z}+ze^{-z}, & z>0 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} ze^{-z}, & z>0 \\ 0, & \text{elsewhere} \end{cases}$$

(c) $F(z) = P(Z \le z)$

$$= P(XY < z)$$

$$= P(Y < z/X)$$



$0<z<1$ $\quad$ $z>1$ $\quad$ $z<0$

Therefore,

$$P(Z \le z) = \begin{cases} 0, & z < 0 \\ \int_0^z \int_0^1 1\, dy\, dx + \int_z^1 \int_0^{z/x} 1\, dy\, dx, & 0 < z < 1 \\ 1, & z \ge 1 \end{cases}$$

$$= \begin{cases} 0, & z < 0 \\ z + \int_z^1 z/x \, dx, & 0 \le z \le 1 \\ 1, & z \ge 1 \end{cases}$$

$$= \begin{cases} 0, & z < 0 \\ z - z\ln(z), & 0 \le z \le 1 \\ 1, & z \ge 1 \end{cases}$$

Consequently,

$$f(z) = \frac{dF}{dz} = \begin{cases} -\ln(z), & 0 \le z \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

#4

A random variable X has a beta distribution of the second kind if, for $\alpha > 0$, $\beta > 0$ its density is

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\beta(\alpha,\beta)(1+x)^{\alpha+\beta}}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability density of the random variable

$$Y = \frac{1}{1+X}.$$

Since $0 \le X < \infty$ it follows that $Y = 1/1+X$ satisfies $0 \le Y \le 1$.
Furthermore,

$$\begin{aligned} G(y) &= P(Y \le y) \\ &= P\left(\frac{1}{1+X} \le y\right) \\ &= P\left(\frac{1}{y} \le 1+X\right) \\ &= P\left(X \ge \frac{1}{y} - 1\right) \\ &= \begin{cases} 0, & y \le 0 \\ \int_{1/y-1}^{\infty} f(x)\,dy, & 0 \le y \le 1 \\ 1, & y \ge 1 \end{cases}. \end{aligned}$$

Therefore,

$$\begin{aligned} g(y) = G'(y) &= \frac{1}{y^2} f\left(\frac{1}{y} - 1\right) \\ &= \frac{1}{y^2} f\left(\frac{1-y}{y}\right) \\ &= \frac{\frac{1}{y^2}\left(\frac{1-y}{y}\right)^{\alpha-1}}{B(\alpha,\beta)\left(\frac{1}{y}\right)^{\alpha+\beta}} \\ &= \frac{(1-y)^{\alpha-1} y^{\beta-1}}{B(\alpha,\beta)}. \end{aligned}$$

Therefore, $Y = 1/1+X$ has a beta distribution. ∎