Probability
Spring 2023
Exam 2
03/24/23

Name (Print): Key

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **You do not have to give explicit numbers for solutions to problems, i.e. you can leave the solution as a product and/or sum of numbers and you do not have to expand out terms with a factorial or binomial coefficients.**
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	12	
2	15	
3	15	
4	10	
5	15	
6	10	
7	15	
8	8	
Total:	100	

Do not write in the table to the right.

1. (12 points) **(Short Answer)** Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

C **I** If X is a continuous random variable with probability density function $p(x)$ then $0 \le p(x) \le 1$.

C **I** If X is a discrete random variable with moment generating function $m(x)$ then $m(0) = 1$.

C **I** If X is a discrete random variable with moment generating function $m(x)$ and $m'(0) = 2$ then $\mu_1 = 0$.

C **I** If X is a discrete random variable then $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X]$.

C **I** If X is a discrete random variable with standard deviation $\sigma > 0$ then

$$\sigma^3 = \mathbb{E}[X^3] - \mathbb{E}[X]^3$$

C **I** If X is a discrete random variable with moments μ_1', μ_2', and μ_2 then

$$\mu_2 = \mu_2' - (\mu_1')^2$$

2. (15 points) A multiple choice exam consists of 5 questions and four answers to each question (of which only one is correct).

   (a) (5 points) If a student answers each question randomly, what is the probability that they will get none of the answers correct?

   $$p = \left(\frac{3}{4}\right)^5$$

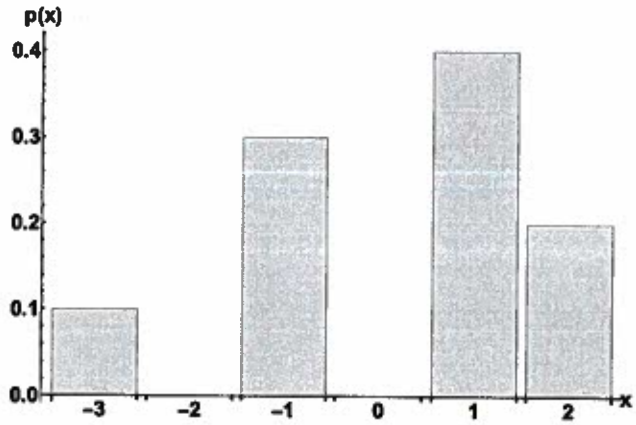
   (b) (5 points) If a passing grade corresponds to getting at least three questions correct, what is the probability that the student will pass the exam?

   $$\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \binom{5}{5}\left(\frac{1}{4}\right)^5$$

   (c) (5 points) What is the probability that they will not pass the exam?

   $$1 - \binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 - \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) - \binom{5}{5}\left(\frac{1}{4}\right)^5$$

3. (15 points) The probability distribution of a discrete random variable X is plotted below as a bar chart.



(a) (5 points) Compute $\mathbb{E}[X]$.

$$\mathbb{E}[X] = -3\cdot\left(\frac{1}{10}\right) - 1\cdot\left(\frac{3}{10}\right) + 1\left(\frac{4}{10}\right) + 2\cdot\left(\frac{2}{10}\right)$$
$$= 2/10$$
$$= 1/5$$

(b) (5 points) Compute $\mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = 9\cdot\left(\frac{1}{10}\right) + 1\cdot\frac{3}{10} + 1\cdot\frac{4}{10} + 4\cdot\frac{2}{10}$$
$$= 24/10$$

(c) (5 points) Compute the standard deviation σ.

$$\sigma = \left(24/10 - 1/25\right)^{1/2}$$
$$= \left(94/50\right)^{1/2}$$
$$= \frac{\sqrt{94}}{2\sqrt{5}}$$

4. (10 points) A box contains 3 balls labeled $1, 2, 3$. From this box one ball is randomly drawn and the number is recorded and then, without replacing the first ball, a second ball is randomly drawn and its number is recorded.

   (a) (5 points) **Short Answer:** Write down the sample space for this experiment.

   $S = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

   (b) (5 points) If X denotes the random variable corresponding to the first recorded number minus the second recorded number, find the probability distribution for X. **You do not have to find an explicit formula, just the probability for all possible values of** X.

   $p(-2) = 1/6$
   $p(-1) = 2/6$
   $p(1) = 2/6$
   $p(2) = 1/6$

5. (15 points) Suppose X is a discrete random variable with the following moment generating function:

$$m(t) = \frac{1}{6}e^{t} + \frac{1}{3}e^{2t} + \frac{1}{2}e^{3t}.$$

(a) (5 points) Using the moment generating function, find μ_1'.

$$\mu_1' = m'(0) = \frac{1}{6}e^{t} + \frac{2}{3}e^{2t} + \frac{3}{2}e^{3t}\Big|_{t=0}$$
$$= \frac{1}{6} + \frac{2}{3} + \frac{3}{2}$$

(b) (5 points) Using the moment generating function, find μ_2'.

$$\mu_2' = m''(0) = \frac{1}{6}e^{t} + \frac{4}{3}e^{2t} + \frac{9}{2}e^{3t}\Big|_{t=0}$$
$$= \frac{1}{6} + \frac{4}{3} + \frac{9}{2}$$

(c) (5 points) Find the probability distribution of X.

$p(1) = 1/6$
$p(2) = 1/3$
$p(3) = 1/2$

6. (10 points) A continuous random variable X has the following probability density

$$p(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \le x < 0 \\ 1-x & 0 \le x < 1 \\ 0 & x > 1 \end{cases}.$$

Find an explicit formula for the cumulative distribution $F(x)$ for this random variable.

$$F(x) = \int_{-\infty}^{x} p(x)\,dx$$

$$= \begin{cases} 0, & x < -1 \\ \int_{-1}^{x}(x+1)\,dx & -1 \le x < 0 \\ \frac{1}{2} + \int_{0}^{x}(1-x)\,dx, & 0 \le x < 1 \\ 1, & x > 1 \end{cases}$$

$$= \begin{cases} 0, & x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2}, & -1 \le x < 0 \\ \frac{1}{2} + x - \frac{x^2}{2}, & 0 \le x < 1 \\ 1, & x > 1 \end{cases}$$

7. (15 points) A continuous random variable X has the following probability density

$$p(x) = \begin{cases} kx^2(2-x) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases},$$

where $k > 0$ is a constant.

(a) (5 points) Find the value of k that makes $p(x)$ a probability density.

$\int_0^2 p(x)dx = 1$

$\Rightarrow \int_0^2 kx^2(2-x)\,dx = 1$

$\Rightarrow k\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2 = 1$

$\Rightarrow k\left(\frac{2\cdot 8}{3} - 4\right) = 1$

$\Rightarrow k = 3/4.$

(b) (5 points) Find $P(X > 1)$.

$P(X>1) = \int_1^\infty p(x)dx$

$= \int_1^2 3/4\, x^2(2-x)dx$

$= 3/4\left(2x^3/3 - \frac{x^4}{4}\right)\Big|_1^2$

$= 3/4\left(\frac{16}{3} - 4 - 2/3 + 1/4\right)$

$= 1 - 1/2 + 3/16 = 16-8+3/16 = 11/16$

(c) (5 points) Find $\mathbb{E}[X]$.

$\mathbb{E}[X] = \int_0^2 kx^3(2-x)dx$

$= k(2x^4/4 - x^5/5)\Big|_0^2$

$= k\left(2\cdot 4 - 32/5\right)$

$= 3/4\,(8/5)$

$= 6/5$

8. (8 points) Let X be a random variable with mean 15 and variance 9. Using Tchebysheff's theorem, find the smallest value of C such that

$$P(|X-15| \geq C) \leq .09$$

$$P(|X-15| \geq k \cdot 3) \leq 1/k^2$$

$$\Rightarrow \frac{1}{k^2} = \frac{9}{100}$$

$$\Rightarrow k = 10/3$$

Therefore, $C = k \cdot \sigma = 10/3 \cdot 3 = 10.$