# MTH 357/657
# Homework #1

Due Date: January 20, 2022

## 1 Problems for Everyone

1. Use Venn diagrams to verify that

   (a) $(A \cap B) \cup (A \cap \bar{B}) = A$

   (b) $(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B) = A \cup B$

2. If $S = \{1, \ldots, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{6, 7, 8, 9\}$, $C = \{2, 4, 8\}$, and $D = \{1, 5, 9\}$, list the elements of the subsets of $S$ corresponding to the following events.

   (a) $\bar{A} \cap B$

   (b) $(\bar{A} \cap B) \cap C$

   (c) $\bar{B} \cup C$

   (d) $(\bar{B} \cup C) \cap D$

   (e) $\bar{A} \cap C$

   (f) $(\bar{A} \cap C) \cap D$

3. Ms. Brown buys one of the houses advertised for sale in Winston Salem, $T$ is the event that the house has three or more baths, $U$ is the event that it has a fireplace, $V$ is the event that it costs more than \$200,000, and $W$ is the event that it is new. Describe in words each of the following events.

   (a) $\bar{T}$, $\bar{U}$, $\bar{V}$, $\bar{W}$

   (b) $T \cap U$, $\bar{U} \cap V$

   (c) $\bar{V} \cup W$, $T \cup U$

4. A coin is tossed once. Then, if it comes up heads, a six sided die is thrown once; if the coin comes up tails, it is tossed twice more. Using the notation in which $(H, 2)$, for example, denotes the event that the coin comes up heads and then the die comes up 2, and $(T, T, H)$ denotes the event that the coin comes up tails then tails and heads, list

   (a) The 10 elements in the sample space $S$.

   (b) The elements of $S$ corresponding to the event $A$ that exactly one head occurs.

   (c) The elements of $S$ corresponding to the event $B$ that at least two tails occur or a number greater than 4 occurs.

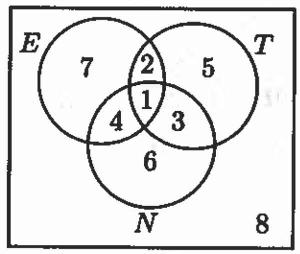
5. An experiment consists of rolling a die until a 3 appears.

   (a) Describe the sample space

   (b) Determine how many elements of the sample space correspond to the event that the 3 appears on the $k$-th roll of the die.

   (c) Describe how many elements of the sample space correspond to the event that the 3 appears not later than the $k$-th roll of the die.

6. In the figure below, $E$, $T$, and $N$ are the events that a car brought to a garage needs an engine overhaul, transmission repairs, or new tires.




   (a) Express in words the events represented by

      i. Region 1

      ii. Region 3

      iii. Regions 1 and 4 together

      iv. Regions 2 and 5 together

      v. Regions 3, 5 and 6 together

   (b) List the region or combination of regions representing the events that a car brought to the garage needs

      i. Transmission repairs, but neither an engine overhaul nor new tires.

      ii. An engine overhaul and transmission repairs.

      iii. Transmission repairs or new tires, but not an engine overhaul.

      iv. New tires.

7. Among 120 visitors to Disneyland, 74 stayed for at least 3 hours, 86 spent at least \$20, 64 went on the Matterhorn ride, 60 stayed for at least 3 hours and spent at least \$20, 52 stayed for at least 3 hours and went on the Matterhorn ride, 54 spent at least \$20 and went on the Matterhorn ride, and 48 stayed for at least 3 hours, spent at least \$20, and went on the Matterhorn ride. Find how many of the 120 visitors to Disneyland

   (a) stayed for at least 3 hours, spent at least \$20, bud did not go on the Matterhorn.

   (b) went on the Matterhorn ride, but stayed less than 3 hours and spent less than \$20.

   (c) stayed less than 3 hours, spent at least \$20, but did not go on the Matterhorn.

   **Hint:** Draw a Venn diagram.

8. Verify the following

   (a) $P(A \cap \bar{B}) = P(A) - P(A \cup B)$

   (b) $P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

9. If $A$ and $B$ are mutually exclusive, $P(A) = .37$, and $P(B) = .44$, find
   (a) $P(\bar{A})$, $P(\bar{B})$
   (b) $P(A \cup B)$
   (c) $P(A \cap B)$, $A \cap \bar{B}$, $\bar{A} \cap \bar{B}$.

10. Given $P(A) = .59$, $P(B) = .30$ and $P(A \cap B) = .21$ find
    (a) $P(A \cup B)$
    (b) $P(A \cap \bar{B})$
    (c) $P(\bar{A} \cup \bar{B})$
    (d) $P(\bar{A} \cap \bar{B})$

11. A hat contains twenty white slips of paper numbered 1 through 20, ten red slips of paper numbered from through 10, forty yellow slips of paper numbered from 1 through 40, and ten blue slips of paper numbered from 1 through 10. If these 80 slips of paper are thoroughly shuffled so that each slip has the same probability of being drawn, find the probabilities of drawing a slip of paper that is
    (a) blue or white
    (b) numbered $1, 2, 4, 4$ or 5
    (c) red or yellow and also numbered $1, 2, 3$ or 4
    (d) numbered 5, 15, 25, or 35
    (e) white and numbered higher than 12 or yellow and numbered higher than 26

12. Four candidates are seeking a vacancy on a school board. If $A$ is twice as likely to be elected as $B$, and $B$ and $C$ are given the same change of being elected, while $C$ is twice as likely to be elected as $D$, what are the probabilities that
    (a) $C$ will win
    (b) $A$ will not win

# 2 Problems for Graduate Students Only

1. Prove the following
   (a) $P(A) \geq P(A \cap B)$
   (b) $P(A) \leq P(A \cup B)$
   (c) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
   (d) $P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

# Homework #1

## #5

An experiment consists of rolling a die until a 3 appears.

(a) Describe the sample space.

(b) Determine how many elements of the sample space correspond to the event that the 3 appears on the $k$-th role of the die.

(c) Describe how many elements of the sample space correspond to the event that the 3 appears not later than the $k$-th role of the die.

## Solution:

(a) $S$ = Set of all finite sequences of the numbers $1, \dots, 6$ that terminate in a 3 and only contain one 3.

(b) Let $E_k$ denote this event and $|E_k|$ the number of elements in $E_k$. Therefore,

$$|E_1| = 1, \ |E_2| = 5, \ |E_3| = 25, \dots .$$

Consequently,

$$|E_k| = 5^{k-1}.$$

(c) Let $F_k$ denote this event and $|F_k|$ the number of elements in $F_k$. Therefore,

$$|F_1| = |E_1| = 1$$

$$|F_2| = |E_1| + |E_2| = 1 + 5 = 6$$

$$|F_3| = |E_1| + |E_2| + |E_3| = 1 + 5 + 25 = 31$$

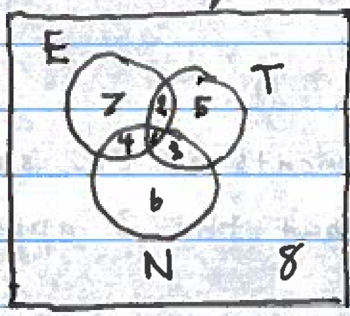
$$\vdots$$

Consequently,

$$|F_k| = \sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k} 5^{i-1} = \frac{1 - 5^k}{1 - 5} = \frac{5^k - 1}{4}. \quad \blacksquare$$

#6.

In the figure below, E, T, and N are the events that a car brought to a car needs an engine overhaul, transmission repairs or new tires




(a) Express in words the events represented by

i) Region 1

ii) Region 3

iii) Regions 1 and 4 together

iv) Regions 2 and 5 together

v) Regions 3, 5, and 6 together

(b) List the region or combination of regions representing the events that a car brought to the garage needs.

i) Transmission repairs, but neither an engine overhaul nor new tires

ii) An engine overhaul and transmission repairs.

iii) Transmission repairs or new tires, but not an engine overhaul.

iv) New tires.

Solutions:

(a) i) The customer needs an engine overhaul, transmission repairs, and new tires.

(ii). The customer needs a new trasmission and new tires but not an engine overhaul.

(iii) The customer needs ~~an~~ an engine overhaul, transmission repairs, and new tires or just an engine overhaul and new ~~transmission~~ tires

(iv) The customer needs a new transmission but not an engine overhaul or new tires or the customer needs a new transmission and an engine overhaul but again does not need new tires.

(v) The customer needs either a new trasmission or new tires or both but in either of these cases does not need an engine overhaul.

(b) (i) Region 5

(ii) Region 2

(iii) Regions 5, 6, 3

(iv) Region 6 1, 3, 4, 6

#8

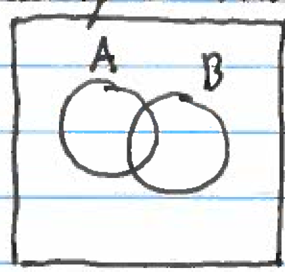
Verify the following

(a) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(b) $P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

~~proof~~

Drawing a Venn diagram helps




(a) Since $A = (A \cap \bar{B}) \cup (A \cap B)$ it follows that

$$P(A) = P((A \cap \bar{B}) \cup (A \cap B))$$
$$= P(A \cap \bar{B}) + P(A \cap B)$$
$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B).$$

(b) Since $S=(\bar{A}\cap\bar{B})\cup(A\cap\bar{B})\cup(\bar{A}\cup B)\cup(A\cap B)$ it follows that

$$P(S)=1=P(\bar{A}\cap\bar{B})+P(A\cap\bar{B})+P(\bar{A}\cup B)+P(A\cap B)$$

$$\Rightarrow 1=P(\bar{A}\cap\bar{B})+P(A\cap\bar{B})+P(A\cap B)+P(\bar{A}\cup B)+P(A\cap B)-P(A\cap B)$$

$$=P(\bar{A}\cap\bar{B})+P(A)+P(B)-P(A\cap B)$$

$$\Rightarrow P(\bar{A}\cap\bar{B})=1-P(A)-P(B)+P(A\cap B)$$

$\blacksquare$

#10

Given $P(A)=.59$, $P(B)=.30$, and $P(A\cap B)=.21$, find

(a) $P(A\cup B)$

(b) $P(A\cap\bar{B})$

(c) $P(\bar{A}\cup\bar{B})$

(d) $P(\bar{A}\cap\bar{B})$.

Solution:

(a) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=.59+.30-.21$

$=.68$

(b) $P(A)=P((A\cap\bar{B})\cup A\cap B)$

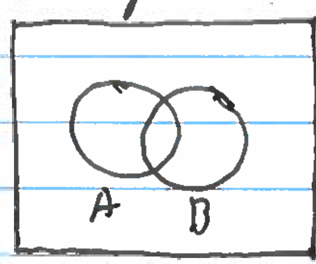
$=P(A\cap\bar{B})+P(A\cap B)$

$\Rightarrow P(A\cap\bar{B})=P(A)-P(A\cap B)$

$=.59-.21$

$=.38$

(c) Drawing a Venn diagram we have




and thus $S=(\bar{A}\cup\bar{B})\cup A\cap B$. Consequently,

$1=P(\bar{A}\cup\bar{B})+P(A\cap B)$

$\Rightarrow 1-.21=P(\bar{A}\cup\bar{B})$

$\Rightarrow P(\bar{A}\cup\bar{B})=.79$.

(d) Since $S=(A\cup B)\cup(\bar{A}\cap\bar{B})$ it follows that

$P(\bar{A}\cap\bar{B})=1-P(A\cup B)$

$\Rightarrow P(\bar{A}\cap\bar{B})=1-.68$

$=.32$

#11

(a) $P(\text{blue or white}) = 3/8$

(b) $P(1,2,3,4 \text{ or } 5) = (5+5+5+5)/80 = 1/4$

(c) $P(\text{red or yellow and also } 1,2,3,4) = (4+4)/80 = 1/8$ 1/10

(d) $P(5, 15, 25 \text{ or } 35) = (2+1+4+1)/80 = 8/80 = 1/8$ 1/10

(e) $P(\text{white and higher than 12 or yellow and higher than 26}) = (8+24)/80 = 2/5$.

11/40

.275