MTH 357/657
Homework #2

Due Date: January 27, 2022

1 Problems for Everyone

1. A person with \$2 in their pocket bets \$1, even money, on the flip of a coin, and they continue to bet \$1 as long as they have money.
   (a) Draw a tree diagram to show the various things that can happen during the first four flips of the coin.
   (b) After the fourth flip of the coin, in how many of the cases will they be exactly even.
   (c) After the fourth flip of the coin, in how many of the cases will they be exactly \$2 ahead.

2 2. Over a thousand years ago a Belgium bishop determined that there are 56 different ways in which three six sided dice can fall provided one is interested only in the overall result and not in which die does what. He assigned a virtue to each of these possibilities and each sinner had to concentrate for some time on the virtue that corresponded to their cast of the dice.
   (a) Find the number of ways in which three dice can all come up with the same number.
   (b) Find the number of ways in which two of the three dice come up with the same number, while the third comes up with a different number.
   (c) Find the number of ways in which all three of the dice come up with different numbers.
   (d) Use the results of (a), (b), (c), and (d) to verify the bishops claim.

3. If the NCAA has applications from six universities for hosting its intercollegiate tennis championships in 2024 and 2025, in how many ways can they select the hosts for these championships
   (a) If they are not both to be held at the same university.
   (b) If they may both be held at the same university.

4. A multiple-choice exam consists of 15 questions, each permitting a choice of three alternatives. In how many different ways can a student check off their answers to these questions.

5. Determine the number of ways in which a distributor can choose two out of fifteen warehouses to ship a larger order to.

1 6. A shipment of 10 television sets includes three that are defective. In how many ways can a hotel purchase four of these sets and receive at least two of the defective sets?

2 7. A baker has five indistinguishable loaves of bread to sell to three customers.

(a) Find the number of ways that the baker can sell all of the loaves of bread to the three customers if it is possible that a customer can not receive any loaves of bread. **Hint:** We might argue that $L|LLL|L$ represents the case where the three customers buy one loaf, three loaves, and one loaf, respectively, and that $LLLL||L$ represents the case where the three customers buy four loaves, none of the loaves, and one loaf. Thus, we must look for the number of ways in which we can arrange the five $L's$ and the two vertical bars.

(b) Find the number of ways that the baker can sell all of the loaves of bread to the three customers so that every customer gets at least one loaf.

2 8. A television director has six time slots to fill with six commercials during an hour "special".

(a) In how many ways cans the director schedule six different commercials during the six time slots if no commercial is to repeat?

(b) In how many ways can the director schedule three different commercials each of which must be shown twice?

(c) In how many ways can the director schedule two different commercials each of which must be shown three times?

1 9. Five people are waiting at a bus stop.

(a) In how many ways can they line up to get on the bus?

(b) In how many ways can they line up if two of the persons refuse to follow each other?

10. Expressing the binomial coefficients in terms of factorials and simplifying algebraically, show that

(a) $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$,

(b) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$,

(c) $n\binom{n-1}{r} = (r+1)\binom{n}{r+1}$.

2 11. Substituting appropriate values for x and y into the binomial formula

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r,$$

where n is a positive integer, show that

(a) $\sum_{r=0}^{n}\binom{n}{r} = 2^n$,

(b) $\cancel{\sum_{r=0}^{n}(-1)^n\binom{n}{r} = 0}$, $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0.$

(c) $\cancel{\sum_{r=0}^{n}(a-1)^r = a^n}$. $\sum_{r=0}^{n}\binom{n}{r}(a-1)^r=a^n$

12. Using the binomial formula

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r,$$

where n is a positive integer, show that

$$\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}.$$

Hint: Differentiate both sides of the binomial formula with respect to y and substitute appropriate values for x and y.

Homework #2.

#2.

Over a thousand years ago a Belgium bishop determined that there are 56 different ways in which three dice can fall provided one is only interested in the overall result and not the order in which each die comes out.

(a) Find the number of ways in which three dice can all come up the same number.

(b) Find the number of ways two numbers match but not the third.

(c) Find the number of ways in which all three come up with different numbers.

(d) Verify the bishops claim.

Solution:

(a) $\binom{6}{1} = 6.$

(b) $\binom{6}{1}\binom{5}{1} = 30$

(c) $\binom{6}{3} = \frac{6\cdot5\cdot4}{3!} = 20$

(d) $6+30+20=56.$

$\blacksquare$

#6.

A shipment of 10 television sets includes three that are defective. In how many ways can a hotel purchase four of these sets and receive at least two of the defective sets?

Solution:

We need to count arrangements of form $w_1 w_2 d_1 d_2$ and $w_1 d_1 d_2 d_3$ assuming w's are equivalent and d's are equivalent. Therefore, the number of ways is given by

$$\# = \frac{4!}{2!2!} + \frac{4!}{3!} = 6 + 4 = 10.$$

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#7

A baker has five indistinguishable loves of bread to sell to three customers.

(a) Find the number of ways that the baker can sell all of the loaves of bread to the three customers assuming it is possible a customer can recieve no loaves.

(b) Find the number of ways that the baker can sell all of the loaves assuming every customer gets at least one loaf.

Solution:

(a) Considering arrangements of the form LLLLL|| we have the number of ways is given by

$$\# = \frac{7!}{5!2!} = \frac{7 \cdot 6}{2} = 21.$$

(b) In this case since three of the loaves are distributed we are interested in arrangements of the form LL|| which gives us

$$\# = \frac{4!}{2!2!} = 6.$$

#8

A television director has six time slots to fill with six commercials.

(a) In how many ways can the director schedule six different commercials during the six time slots if no commercial is to repeat?

(b) In how many ways can the director schedule three different commercials each of which must be shown twice?

(c) In how many ways can the director schedule two different commercials each of which must be shown three times?

Solution:

(a) Since each commercial is different, the number of ways is given by

$$\# = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 6! = 720$$

(b) In this case we care about arrangements of the form $a_1a_2b_1b_2c_1c_2$ where a's, b's, c's are indistinguishable. Therefore, number of ways is given by

$$\# = \frac{6!}{2!2!2!} = \frac{6\cdot5\cdot3\cdot2}{2} = 6\cdot5\cdot3 = 90.$$

(c) In this case we care about arrangements of the form $a_1a_2a_3b_1b_2b_3$ where a's, b's, c's are indistinguishable. Therefore, number of ways is given by

$$\# = \frac{6!}{3!3!} = \frac{6\cdot 5\cdot 4}{6} = 20.$$

#9.

Five people are waiting at a bus stop.

(a) In how many ways can they line up to get on the bus?

(b) In how many ways can they line up if two of the persons refuse to follow each other?

Solution:

(a) Here, order matters and thus we have $5! = \cancel{125}$ 120 ways.

(b). If two are next to each other than there are $4!$ ways to arrange them. Therefore, if we want them to be not next to each other, the number of ways is given by

$$\# = 5! - 4! = 4!(5-1) = 4\cdot 4! = 96 \text{ ways.}$$

or 72

#11.

Show that

(a) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

(b) $\sum_{r=0}^{n} (-1)^n \binom{n}{r} = 0$

(c) $\sum_{n=0}^{n} (a-1)^r = a^n$

Solution:

(a) Since $(x+y)^n = \sum_{r \geq 0}^{n} \binom{n}{r} x^{n-r} y^r$, it follows that if $x=1, y=1$ then

$$(1+1)^n = 2^n = \sum_{r \geq 0}^{n} \binom{n}{r} (1)^{n-r} 1^r = \sum_{r \geq 0}^{n} \binom{n}{r}.$$

(b) Letting $x=1, y=-1$ we have

$$(1-1)^n = 0 = \sum_{r \geq 0}^{n} \binom{n}{r} (1)^{n-r} (-1)^r = \sum_{r \geq 0}^{n} \binom{n}{r} (-1)^r.$$

(c) Letting $x=1$ and $y=a-1$ we have that

$$(1+a-1)^n = a^n = \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (a-1)^r = \sum_{r=0}^{n} \binom{n}{r} (a-1)^r.$$

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