MTH 357/657
Homework #5

Due Date: February 24, 2023

1 Probability Distributions

1. For each of the following, determine whether the given function can serve as the probability distribution of a random variable with the given range

   (a) $f(x) = \dfrac{x-2}{5}$ for $x = 1, 2, 3, 4, 5$.

   (b) $f(x) = \dfrac{x^2}{30}$ for $x = 0, 1, 2, 3, 4$.

   (c) $f(x) = \dfrac{1}{5}$ for $x = 0, 1, 2, 3, 4, 5$.

   (d) $f(x) = \dfrac{2x}{k(k+1)}$ for $x = 1, 2, 3 \ldots, k$.

2. For each of the following, determine c so that the function can serve as the probability distribution of a random variable with the given range:

   (a) $f(x) = cx$ for $x = 1, 2, 3, 4, 5$;

   (b) $f(x) = c\left(\dfrac{5}{x}\right)$ for $x = 0, 1, 2, 3, 4,$;

   (c) $f(x) = c\left(\dfrac{1}{4}\right)^x$ for $x = 1, 2, 3, \ldots$.

3. Find the probability distribution of X, the difference between the number of heads and the number of tails obtained in four tosses of a balanced coin.

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4. An urn contains four balls numbered $1, 2, 3$, and 4. If two balls are drawn from the urn at random find the probability distributions for the following:

   (a) The largest of the two sampled numbers;

   (b) The sum of the two sampled numbers.

5. A coin is biased so that heads is twice as likely as tails. For three independent coin tosses of the coin, find the probability distribution of X the total number of heads.

6. pg. 91, #3.9, 3.11

2 Expected Values

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1. pg. 97, #3.12, pg. 100, #3.34.

2 2. Find the expected value of the random variable X having the probability distribution

$$f(x) = \frac{|x-2|}{7} \text{ for } x = -1, 0, 1, 3.$$

3. A game of chance is considered fair or equitable, if each player's expectation is equal to zero. If someone pays us \$10 each time that we roll a 3 or a 4 with a balanced die, how much should we pay that person when we roll a 1, 2, 5, 6 to make the game equitable.

4. The manager of a bakery knows that the number of chocolate cakes they can sell on any given day is a random variable with the probability distribution $f(x) = 1/6$ for $x = 0, 1, 2, 3, 4, 5$. They also know there is a profit of \$1.00 for each cake they sell and a loss of .40 for each cake they do not sell. Assuming that each cake can only be sold on the day it is made, find the baker's expected profit for a day in which they bake
   (a) one cake;
   (b) two cakes;
   (c) three cakes;
   (d) four cakes;
   (e) five cakes.

3 Bernoulli type distributions

1. pg. 111, #3.39, 3.44 – 3.48 2
2. pg. 119, #3.70, 3.71, 3.76. 2
3. A shipment of 80 burglar alarms contains four that are defective. If three from the shipment are randomly selected and shipped to a customer, find the probability that the customer will get exactly one bad unit using
   (a) the hypergeometric distribution;
   (b) the binomial distribution as an approximation.
4. A panel of 300 persons chosen for jury duty includes 30 individuals under 25 years of age. Since this jury of 12 persons chosen from this panel does not include anyone under 25 years of age, the defense attorney complained that this jury is not really representative. Indeed, the attorney argued, if the selection were random, the probability of having one of the 12 jurors under 25 years of age should be *many times* the probability of having none of them under 25 years of age. Actually, what is the ratio of these two probabilities?

4 Poisson Distribution

1. It is known from experience that 1.4 percent of the calls received by a switchboard are wrong numbers. Use the Poisson distribution approximation to the binomial distribution to find the probability that among 150 calls received by the switchboard two are wrong numbers.

2. Records show that the probability is .0012 that a person will get food poisoning spending a day at a certain state fair. Use the Poisson approximation to the binomial distribution to determine the probability that among 1000 persons attending the fair at most two will get food poisoning.

3. pg. 136, #3.123, 3.127, 3.128.

Homework #5

#4.

An urn contains four balls numbered 1, 2, 3, and 4. If two balls are drawn from the urn at random find the probability distribution for the following

(a) The largest of the two sampled numbers

(b) The sum of the two sampled numbers.

Solution:

The sample space is $S = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$.

(a) If $X: S \to \mathbb{R}$ is defined by $X((x,y)) = \max\{x,y\}$ then $p(x)$ satisfies

$p(1) = 0$

$p(2) = 1/6$

$p(3) = 2/6$

$p(4) = 3/6$.

(b) If $X: S \to \mathbb{R}$ is defined by $X(x,y) = x+y$ then

$p(1) = 0$

$p(2) = 0$

$p(3) = 1/6$

$p(4) = 1/6$

$p(5) = 2/6$

$p(6) = 1/6$

$p(7) = 1/6$.

$\blacksquare$

pg. 97, #3.12

Let Y be a random variable with $p(y)$ given by

$$p(1) = .4,$$
$$p(2) = .3,$$
$$p(3) = .2,$$
$$p(4) = .1.$$

Find $\mathbb{E}[Y]$, $\mathbb{E}[1/Y]$, $\mathbb{E}[Y^2-1]$, and $V(Y)$.

Solution:

$$\mathbb{E}[Y] = .4 + 2 \cdot .3 + 3 \cdot .2 + 4 \cdot .1 = 2$$

$$\mathbb{E}[1/Y] = .4 + \frac{1}{2} \cdot .3 + \frac{1}{3} \cdot .2 + \frac{1}{4} \cdot .1 = .7567$$ v42

$$\mathbb{E}[Y^2-1] = \mathbb{E}[Y^2] - 1 = 1 \cdot .4 + 4 \cdot .3 + 9 \cdot .2 + 16 \cdot .1 - 1 = 4$$

$$V[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 3 - 2 = 1.$$

5 - 4 ? = 1

#2.

Find the expected value of a random variable X having the distribution

$$f(x) = \frac{|x-2|}{7} \quad \text{for} \quad x = -1, 0, 1, 3$$

Solution:

$$\mathbb{E}[X] = \frac{3}{7} \cdot (-1) + 0 \cdot \frac{2}{7} + \frac{1}{7} \cdot 1 + \frac{1}{7} \cdot 3$$

$$= 1/7$$

pg. 112, #3.48

A missile protection system consists of n radar sets operating independently, each with a probability of .9 of detecting a missile.

(a) If $n=5$ what is the probability that exactly four sets detect the missile? At least one set?

(b) How large must n be if we require the probability of detecting a single missile to be .999.

Solution:

(a) $p_1 = \binom{5}{4}(.9)^4(.1) = .3281$

$$p_2 = 1 - \binom{5}{0}(.1)^5 = 1 - \left(\frac{1}{10}\right)^5 = .99999$$

(b) $p = 1 - \binom{n}{0}(.1)^n = .999$

$$\Rightarrow (.1)^n = .001$$

$$\Rightarrow \left(\frac{1}{10}\right)^n = \left(\frac{1}{10}\right)^3$$

$$\Rightarrow n = 3.$$

$\blacksquare$

pg. 119, #3.76

If Y has a geometric distribution with success probability (.3), what is the largest y_0 such that $P(Y \geq y_0) \geq .1$?

Solution:

$$P(Y \geq y_0) = \sum_{y=y_0}^{\infty} (.7)^{y-1}(.3) = \frac{.3}{.7}\sum_{y=y_0}^{\infty}(.7)^y = \frac{3}{7}\sum_{y=0}^{\infty}(.7)^{y+y_0}$$

Therefore,

$$P(Y \geq y_0) = \tfrac{3}{7}(.7)^{y_0} \sum_{y=0}^{\infty} (.7)^y = \tfrac{3}{7}(.7)^{y_0} \frac{1}{1-\tfrac{7}{10}} = \frac{10}{7}\left(\frac{7}{10}\right)^{y_0}$$

Consequently, we need

$$\frac{10}{7}\left(\frac{7}{10}\right)^{y_0} = .1$$

$$\Rightarrow y_0 \ln\left(\frac{7}{10}\right) = \ln\left(\frac{7}{100}\right)$$

$$\Rightarrow y_0 = \frac{\ln(7/100)}{\ln(7/10)} = 7.4557.$$

Since $y_0 \in \mathbb{N}$ it follows that we need $y_0 \geq 8$.

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