MTH 357/657
Homework #8

Due Date: March 31, 2023

1 Moment Generating Functions

1. Suppose a and b are constants and X is a continuous random variable with moment generating function $m_X(t)$ and let $Y = X + a$, $Z = bX$, and $U = (X + a)/b$ be random variables with moment generating functions $m_Y(t)$, $m_Z(t)$, and $m_U(t)$ respectively. Show that the following identities are true:

   (a) $m_Y(t) = e^{at} m_X(t)$,

   (b) $m_Z(t) = m_X(bt)$,

   (c) $m_U(t) = e^{a/bt} m_X(t/b)$.

2 Gamma Distributions

1. Find the probabilities that the value of a random variable will exceed 4 if it has a gamma distribution with

   (a) $\alpha = 2$ and $\beta = 3$;

   (b) $\alpha = 3$ and $\beta = 4$.

2. Show that a gamma distribution with $\alpha > 1$ has a relative maximum at $x = \beta(\alpha - 1)$. What happens when $0 < \alpha < 1$ and when $\alpha = 1$?

3. Show that for $t < 1/\beta$ the moment generating function of the gamma distribution is given by

$$m(t) = (1 - \beta t)^{-\alpha}.$$

   Hint: Make the substitution $u = x(1/\beta - t)$ in the integral defining $m(t)$.

4. A random variable X has a Weibull distribution if and only if its probability density is given by

$$p(x) = \begin{cases} kx^{\beta-1}e^{-\alpha x^{\beta}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases},$$

   where $\alpha > 0$ and $\beta > 0$.

   (a) Express k in terms of α and β.

   (b) Show that

$$\mu = \mathbb{E}[X] = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right).$$

5. If X is a random variable with an exponential distribution, i.e., a gamma distribution with $\alpha = 1$, and $P(X > 2) = .08$, determine the value of β and then compute $P(X \leq 1.5)$.
6. Suppose X is an exponential distribution with $\mu = \mathbb{E}[X] = 10$. Find the mean and variance of the following random variable:

$$C = 100 + 40X + 3X^2.$$

3 Beta Distributions

1. The percentage of impurities per batch in a chemical product is a random variable X with probability density function

$$p(x) = \begin{cases} kx^2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

   A batch with more than 40% impurities cannot be sold.

   (a) Find the value of k that makes this a probability density.

   (b) Determine the probability that a randomly selected batch cannot be sold because of excessive impurities.

   (c) Find the mean and variance of the of the percentage of impurities in a randomly selected batch of the chemical.

2. Prove that the variance of a beta-distributed random variable with parameters α and β is

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

3. For $\alpha, \beta > 1$, find the local maximum for the probability density function for the beta distribution in terms of α and β.

4 Normal Distribution

1. Show that a normal distribution has

   (a) a relative maximum at $x = \mu$;

   (b) inflection points at $x = \mu - \sigma$ and $x = \mu + \sigma$.

2. Assume that X is normally distributed with mean μ and standard deviation σ. After observing a value of X, a mathematician constructs a rectangle with length $L = |Y|$ and width $W = 3|Y|$. If A denotes the area of the resulting rectangle, calculate $\mathbb{E}(A)$.