Lecture 12: Poisson Distribution

Recall:

Binomial Distribution:

$$p(x) = \binom{n}{x} p^x q^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}$$

How to approximate when n is large and p is small.
For example, radioactive decay:

- Radium has a half life of 1600 years.
- 1 mol has 10^{23} atoms.
- X = number of decayed atoms after 1 second
- p decay in time Δt years is
  $1 - 2^{-\Delta t/1600}$

$$\Rightarrow p \text{ in 1 minute} = 1 - 2e^{-3 \cdot 17 \times 10^{-8}/1600} = 1.37 \times 10^{-11}$$

$$\Rightarrow P(X = x) = \binom{10^{23}}{x} (1.37 \times 10^{-11})^x (1 - 1.37 \times 10^{-11})^{10^{23} - x}$$

(Completely intractable calculation).

Poisson Distribution

Introduce a parameter $\boxed{\lambda = np}$ assume $n \to \infty$ and $p \to 0$.

$$\Rightarrow p(x) = \binom{n}{x} p^x (1-p)^{1-x}$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{x!\, n^x} \lambda^x \left[\left(1 - \frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{(1 - 1/n)\cdots(1 - x/n + 1/n)}{x!} \lambda^x \left[\left(1 - \frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

Letting $n \to \infty$ we have that

1. $\lim_{n\to\infty} (1-1/n)\cdots(1-x/n+1/n)=1$

2. $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-n/\lambda}=e.$

3. $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}=1$

$$\Rightarrow \lim_{n\to\infty} p(x)=\frac{\lambda^x e^{-\lambda}}{x!}$$

Definition - A random variable X has a Poisson distribution if and only if

$$p(x)=\frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x=0,1,2,\ldots$$

proof:

1. $p(x)\geq 0$
2. $\sum_x p(x)=\sum_x \lambda^x/x!\,e^{-\lambda} = e^{-\lambda}\sum_x \lambda^x/x! = e^{-\lambda}\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\ldots\right)$

$$\Rightarrow \sum_x p(x)=e^{-\lambda}e^{\lambda}=1.$$

Theorem - For a Poisson distribution $\mathbb{E}[X]=\lambda.$

proof:

$$\mathbb{E}[X]=\sum_x x\lambda^x e^{-\lambda}/x! = e^{-\lambda}\sum_x \lambda^x/(x-1)! = e^{-\lambda}(\lambda^1+\lambda^2/1!+\lambda^3/2!+\ldots)$$

$$\Rightarrow \mathbb{E}[X]=e^{-\lambda}\cdot\lambda e^{\lambda}=\lambda.$$

Example:

2% of books bound have defective bindings. What is the approximate probability that 5 of 400 books will have defective bindings

$$\lambda = p \cdot n = (.02) \cdot (400) = 8$$

$$\Rightarrow p(5) = \frac{8^5 e^{-8}}{5!} = .093.$$

Example:

A certain sheet metal has, on average, five defects per 10 square feet. If we assume a poisson distribution, What is the probability that a 15-square sheet of metal will have at least six defects.

$\rightarrow$ For 10 sq. feet $\lambda_1 = 5$

$\rightarrow$ For 15 sq. feet $\lambda_2 = 5 \cdot 15/10 = 5 \cdot 3/2 = 7.5$

$\rightarrow P(X \geq 6) = 1 - P(X \leq 5) = \sum_{x=0}^{5} 7.5^x e^{-7.5}/x! = .1550.$