Lecture 13: Moment Generating Functions

Definition - Let X be a random variable

1. $\mu_k' = \mathbb{E}[X^k]$ is the k-th moment about the origin.
2. $\mu_k = \mathbb{E}((X-\mu)^k)$ is the k-th moment about the mean.

Definition - The moment generating function $m(t)$ for a random variable X is defined to be

$$m(t) = \mathbb{E}[e^{tX}]$$
$$= \mathbb{E}[1 + tX + \tfrac{1}{2}t^2X^2 + \tfrac{t^3}{3!}X^3 + \dots]$$
$$= 1 + t\mu_1' + \tfrac{t^2}{2}\mu_2' + \tfrac{t^3}{3!}\mu_3' + \dots$$
$$\Rightarrow \left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = m^{(k)}(0) = \mu_k'.$$

Therefore, if you find a moment generating function you can easily calculate the mean and variance.

Example:

Suppose X is a Poisson random variable with

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$
$$\Rightarrow \mathbb{E}[e^{tX}] = \sum_x \frac{e^{tx}\lambda^x e^{-\lambda}}{x!}$$
$$= e^{-\lambda}\sum_x \frac{(e^t\lambda)^x}{x!}$$
$$= e^{-\lambda}e^{\lambda e^t}$$
$$= e^{\lambda(e^t-1)}.$$
$$\Rightarrow \mathbb{E}[X] = \left.\frac{d}{dt}e^{\lambda(e^t-1)}\right|_{t=0} = \left.\lambda e^t e^{\lambda(e^t-1)}\right|_{t=0} = \lambda$$
$$\mathbb{E}[X^2] = \left.\frac{d^2}{dt^2}e^{\lambda(e^t-1)}\right|_{t=0} = \left.\lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t}e^{\lambda(e^t-1)}\right|_{t=0}$$

$$\Rightarrow \mathbb{E}[X^2] = \lambda^2 + \lambda$$

Therefore, $\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.

Example:

Let X have a binomial distribution with n-trials

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

$$\Rightarrow \mathbb{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

$$= m(t)$$

Therefore,

$$m'(t) = n(pe^t + q)^{n-1} pe^t$$

$$\Rightarrow m'(0) = n(p+q)^{n-1} p = np = \mu_1'$$

$$m''(t) = n(n-1)(pe^t + q)^{n-2} p^2 e^{2t} + n(pe^t + q)^{n-1} pe^t$$

$$\Rightarrow m''(0) = n(n-1)p^2 + np$$

$$\Rightarrow \sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = n(n-1)p^2$$

Example:

Let X be the geometric distribution with

$$p(x) = pq^{x-1}$$

$$\Rightarrow \mathbb{E}[e^{tX}] = \sum_{x=1}^{\infty} pq^{x-1}e^{tx}$$

$$= pq^{-1}\sum_{x=1}^{\infty}(qe^t)^x$$

$$= pq^{-1}\left(\frac{1}{1-qe^t} - 1\right)$$

$$= pq^{-1}\left(\frac{qe^t}{1-qe^t}\right)$$

$$= \frac{pe^t}{1-qe^t}.$$

$$= m(t)$$

$$\Rightarrow m'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t}{(1-qe^t)^2}$$

$$\Rightarrow \mu_1' = m'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}.$$

$$\Rightarrow m''(t) = \frac{(1-qe^t)^2pe^t - pe^t2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{pe^t(1-qe^t+2qe^t)}{(1-qe^t)^3}$$

$$= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$\Rightarrow m''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2} = \frac{1}{p^2} + \frac{1-p}{p^2}$$

Therefore,

$$\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1-p}{p^2}.$$