Lecture 16: Continuous Random Variables

Example:

Want to know probability that an accident occurs at a location on a 200 km freeway assuming each is equally likely

$$0 \qquad\qquad 200$$

$S = \{x : 0 \le x \le 200\}$.

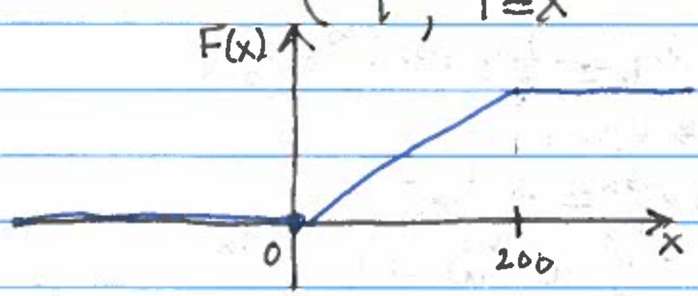
$\Rightarrow P(A) = \dfrac{L(A)}{200}$ works if A is an interval.

1. $P(A) \ge 0$.
2. $P(S) = \dfrac{L(S)}{200} = 1$.
3. If $A_1, A_2, \ldots$ are disjoint then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup \ldots) &= L(A_1 \cup A_2 \cup A_3 \cup \ldots)/200 \\ &= \frac{L(A_1) + L(A_2) + \ldots}{200} \\ &= \frac{L(A_1)}{200} + \frac{L(A_2)}{200} + \ldots \\ &= P(A_1) + P(A_2) + \ldots \end{aligned}$$

$\Rightarrow$ The CDF is given by

$$F(x) = P(X \le x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{200}, & 0 \le x \le 200 \\ 1, & 1 \le x \end{cases}$$



Note, $P(0 \le a \le x \le b) = F(b) - F(a) = \frac{b}{200} - \frac{a}{200} = \frac{b-a}{200}$.

Also, by the fundamental theorem of calculus:

$$F(b) - F(a) = \int_a^b \frac{dF}{dx}\,dx = \int_a^b \frac{1}{200}\,dx$$

The function

$$\boxed{f(x) = \frac{dF}{dx}}$$

is called the probability density function.

Theorem - $P(a \le X \le b) = \int_a^b f(x)\,dx$.

Theorem - A function can serve as a probability density function of a continuous random variable X if its values, $f(x)$ satisfy

1. $f(x) \ge 0$, $-\infty < x < \infty$;
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Example:

If X has the probability density

$$f(x) = \begin{cases} Ke^{-3x}, & x > 0 \\ 0 & x \le 0 \end{cases}$$

1. find K;
2. find $P(1 \le X \le 2)$
3. $F(x)$

1. $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} Ke^{-3x}\,dx = -\frac{1}{3}Ke^{-3x}\Big|_0^{\infty} = \frac{1}{3}K$
   $\Rightarrow \frac{1}{3}K = 1 \Rightarrow K = 3$
2. $P(1 \le X \le 2) = \int_1^2 3e^{-3x} = e^{-3} - e^{-6}$
3. $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 3e^{-3x}\,dx = 1 - e^{-3x}$.

Definition - The expected value of a random variable X with continuous density $f(x)$ is given by

- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$
- $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$

Example:

1. $f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\Rightarrow \mathbb{E}[X] = \int_0^{\infty} 3xe^{-3x}\,dx$$
$$= -xe^{-3x}\Big|_0^{\infty} + \int_0^{\infty} e^{-3x}\,dx$$
$$= 0 - \frac{1}{3}e^{-3x}\Big|_0^{\infty} = 1/3.$$

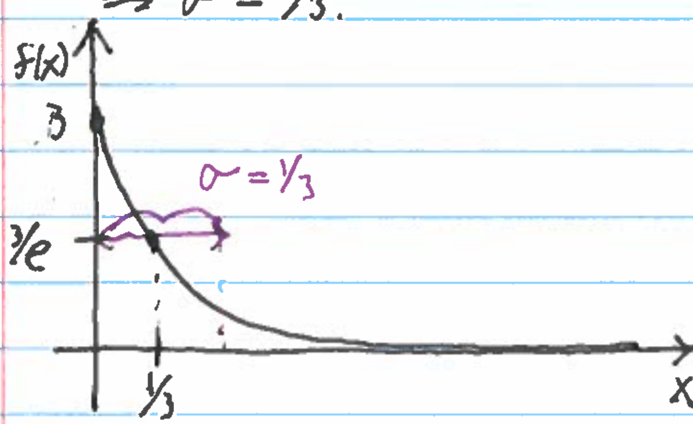
$$\Rightarrow \mathbb{E}[X^2] = \int_0^{\infty} 3x^2 e^{-3x}\,dx$$

Let $u = 3x \Rightarrow du = 3dx,\ x = u/3$

$$\mathbb{E}[X^2] = \frac{1}{3}\int_0^{\infty} 3\,u^2/9\, e^{-u}\,du$$
$$= \frac{1}{9}\int_0^{\infty} u^2 e^{-u}\,du$$
$$= \frac{1}{9}\left(-\cancel{u^2 e^{-u}}\Big|_0^{\infty} + \int_0^{\infty} 2ue^{-u}\,du\right)$$
$$= \frac{1}{9}\left(-2ue^{-u}\Big|_0^{\infty} + \int_0^{\infty} 2e^{-u}\,du\right)$$
$$= 2/9$$

Therefore,

$$\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2/9 - 1/9 = 1/9$$
$$\Rightarrow \sigma = 1/3.$$



2. $f(x) = \begin{cases} 2/x^2, & 0 \le x < \infty \\ 0, & x < 0 \end{cases}$

$\Rightarrow \mathbb{E}[X] = \int_0^\infty x \, 2/x^2 \, dx = 2 \int_0^\infty \frac{1}{x} dx = \infty$