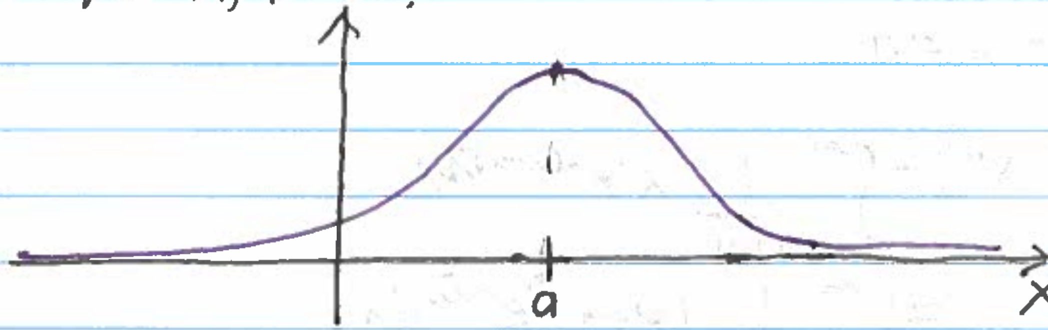
Lecture 19: The normal Probality Distribution.

Definition - A random variable X has a normal or Gaussian distribution if and only if the density function is given by

$$p(x) = K e^{-(x-a)^2/b},$$

$K > 0$, $a \in \mathbb{R}$, $b > 0$.



Properties:

1. $$\int_{-\infty}^{\infty} p(x)dx = K\int_{-\infty}^{\infty} e^{-(x-a)^2/b}dx \quad (u = x-a)$$

$$= K\int_{-\infty}^{\infty} e^{-u^2/b}du \quad (v = u/\sqrt{b} \Rightarrow \sqrt{b}\,dv = du)$$

$$= K\sqrt{b}\int_{-\infty}^{\infty} e^{-v^2}dv$$

$$= 2K\sqrt{b}\int_{0}^{\infty} e^{-v^2}dv \quad (z = v^2 \Rightarrow dz = 2v\,dv)$$

$$= K\sqrt{b}\int_{0}^{\infty} z^{-1/2}e^{-z}dz$$

$$= K\sqrt{b}\,\Gamma(1/2)$$

$$= K\sqrt{b\pi}$$

Therefore,

$$K = \frac{1}{\sqrt{b\pi}}.$$

2. $\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{b\pi}} x e^{-(x-a)^2/b} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{b\pi}} (u+a) e^{-u^2/b} du$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{b\pi}} u e^{-u^2/b} du + \int_{-\infty}^{\infty} \frac{a}{\sqrt{b\pi}} e^{-u^2/b} du$

$= 0 + a$

$\Rightarrow \mu = a.$

3. $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{b\pi}} x^2 e^{-(x-a)^2/b} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{b\pi}} (u+a)^2 e^{-u^2/b} dx$

$= 2\int_{0}^{\infty} \frac{1}{\sqrt{b\pi}} u^2 e^{-u^2/b} + a^2 \quad (v = u^2/b \Rightarrow dv = 2u/b)$

$= \int_{0}^{\infty} \sqrt{\frac{b}{\pi}} \cdot v^{1/2} b^{1/2} e^{-v} dv + a^2$

$= \frac{b}{\sqrt{\pi}} \int_{0}^{\infty} v^{1/2} e^{-v} dv + a^2$

$= \frac{b}{\sqrt{\pi}} \Gamma(3/2) + a^2$

$= \frac{b}{\sqrt{\pi}} \frac{1}{2} \Gamma(1/2) + a^2$

$= \frac{b}{2} + a^2$

3. $\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = b/2$

4. Putting it all together we have:

$$\boxed{p(x) = N(\mu, \sigma^2)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$$

Definition: The standard normal distribution has the probability density:

$$N(0,1)(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Example:

Suppose $a < \mu$ and $b > \mu$, then

$$P(a < N(\mu,\sigma) < b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{a-\mu}^{b-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} du \quad (v = u/\sigma)$$

$$= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$$= P\left(\frac{a-\mu}{\sigma} < N(0,1) < \frac{b-\mu}{\sigma}\right).$$