Lecture 24: Multivariate Moments.

Definition - X, Y are continuous random variables with density $p(x,y)$.

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,p(x,y)\,dy\,dx.$$

Example:

Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y), & 0<x<1,\ 1<y<2 \\ 0, & \text{e.w.} \end{cases}$$

find the expected value of $g(X,Y) = X/Y^3$.

$$\Rightarrow \mathbb{E}[X/Y^3] = \int_1^2\int_0^1 \frac{2x(x+2y)\,dx\,dy}{7y^3}$$

$$= \int_1^2\int_0^1 \frac{2}{7}\left(\frac{x^2+2xy}{y^3}\right)dx\,dy$$

$$= \int_1^2 \frac{2}{7}\left(\frac{x^3}{3y^3} + \frac{x^2y}{y^3}\right)\Bigg|_0^1 dy$$

$$= \int_1^2 \frac{2}{7}\left(\frac{1}{3y^3} + \frac{1}{y^2}\right)dy$$

$$= \frac{15}{84}.$$

Definition - Let X, Y be continuous random variables with means μ_X, μ_Y.

$$Cov(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)]$$

and measures the association between X and Y.

~ X, Y both above or below mean $\Rightarrow Cov(X,Y) > 0$

~ X, Y opposite signs $\Rightarrow Cov(X,Y) < 0$.

Theorem - $Cov(X,Y) = \mathbb{E}[XY] - \mu_X \mu_Y$.

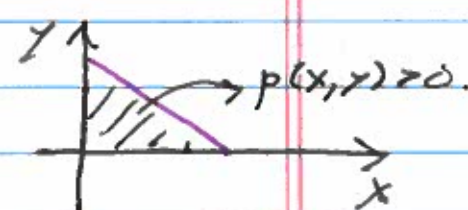
proof:

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathbb{E}[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &= \mathbb{E}[XY] - \mu_X \mathbb{E}[Y] - \mu_Y \mathbb{E}[Y] + \mu_X \mu_Y \mathbb{E}[1] \\ &= \mathbb{E}[XY] - \mu_X \mu_Y. \end{aligned}$$

Example:

Find the covariance of the two random variables X, Y whose joint density is given by

$$f(x,y) = \begin{cases} 2, & x>0,\ y>0,\ x+y<1 \\ 0, & \text{o.w.} \end{cases}$$



$$\mu_X = \mathbb{E}[X] = \int_0^1 \int_0^{1-x} 2x \, dy \, dx = 1/3$$

$$\mu_Y = \mathbb{E}[Y] = \int_0^1 \int_0^{1-x} 2y \, dy \, dx = 1/3$$

$$\mathbb{E}[XY] = \int_0^1 \int_0^{1-x} 2xy \, dy \, dx = 1/12$$

$$\Rightarrow Cov(X,Y) = 1/12 - 1/9 = -1/36$$

Theorem - If X and Y are independent then.

1. $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
2. $\mathrm{Cov}(X,Y) = 0$

proof:

1. $$\begin{aligned}\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dy\, dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x) g(y)\, dy\, dx \\ &= \int_{-\infty}^{\infty} x f(x) \left(\int_{-\infty}^{\infty} y\, g(y)\, dy\right) dx \\ &= \int_{-\infty}^{\infty} x f(x)\, \mu_Y\, dx \\ &= \mu_Y \mu_X.\end{aligned}$$

2. $$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[XY] - \mu_X \mu_Y \\ &= \mathbb{E}[X]\mathbb{E}[Y] - \mu_X \mu_Y \\ &= 0.\end{aligned}$$

Theorems -

1. $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
2. $\mathrm{Cov}(X,Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$
3. $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$.

proof:

1. $$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathbb{E}[(Y-\mu_Y)(X-\mu_X)] \\ &= \mathrm{Cov}(Y,X).\end{aligned}$$

2. $$\begin{aligned}\mathrm{Cov}(X,Y+Z) &= \mathbb{E}[(X-\mu_X)(Y+Z-\mu_Y-\mu_Z)] \\ &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y) + (X-\mu_X)(Z-\mu_Z)] \\ &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] + \mathbb{E}[(X-\mu_X)(Z-\mu_Z)] \\ &= \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z).\end{aligned}$$

3. $$\begin{aligned}\mathrm{Cov}(aX,Y) &= \mathbb{E}[(aX-a\mu_X)(Y-\mu_Y)] \\ &= a\,\mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= a\,\mathrm{Cov}(X,Y).\end{aligned}$$

Example:

X, Y, Z are continuous random variables

- $\mu_X = 2,\ \mu_Y = -3,\ \mu_Z = 4$
- $\sigma_X^2 = 1,\ \sigma_Y^2 = 5,\ \sigma_Z^2 = 2$
- $Cov(X,Y) = -2,\ Cov(X,Z) = -1,\ Cov(Y,Z) = 1.$

If $W = 3X - Y + 2Z$ find μ_W, σ_W^2.

- $$\begin{aligned} \mathbb{E}[W] &= \mathbb{E}[3X - Y + 2Z] \\ &= 3\mathbb{E}[X] - \mathbb{E}[Y] + 2\mathbb{E}[Z] \\ &= 3\cdot 2 + 3 + 2\cdot 4 \\ &= 17 \end{aligned}$$

- $$\begin{aligned} \sigma_W^2 &= Cov(W,W) \\ &= Cov(3X - Y + 2Z, 3X - Y + 2Z) \\ &= Cov(3X - Y + 2Z, 3X) + Cov(3X - Y + 2Z, -Y) + Cov(3X - Y + 2Z, 2Z) \\ &= 3Cov(3X - Y + 2Z, X) - Cov(3X - Y + 2Z, Y) + 2Cov(3X - Y + 2Z, Z) \\ &= 3(3Cov(X,X) - Cov(Y,X) + 2Cov(Z,X)) \\ &\quad - (3Cov(X,Y) - Cov(Y,Y) + 2Cov(Z,Y)) \\ &\quad + 2(3Cov(X,Z) - Cov(Y,Z) + 2Cov(Z,Z)) \\ &= 3(3\cdot 1 + 2 + 2\cdot 1) - (3\cdot(-2) - 5 + 2\cdot 1) + 2(3\cdot(-1) - (-2) + 2\cdot 2)) \\ &= 3\cdot 7 + 9 + 2(3) \\ &= 36. \end{aligned}$$

Theorem - If $X_1, \ldots, X_n$ are independent and $Z = \sum a_i X_i$ then

$$\sigma_Z^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2.$$

proof:

$$\sigma_Z^2 = Cov(Z,Z) = Cov\left(\sum_{i=1}^{N} a_i X_i, \sum_{j=1}^{N} a_j X_j\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j Cov(X_i, X_j)$$

$$\Rightarrow \sigma_Z^2 = \sum_{i=1}^{N} a_i^2 Cov(X_i, X_i) = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2.$$