Lecture 26: Transformation Technique

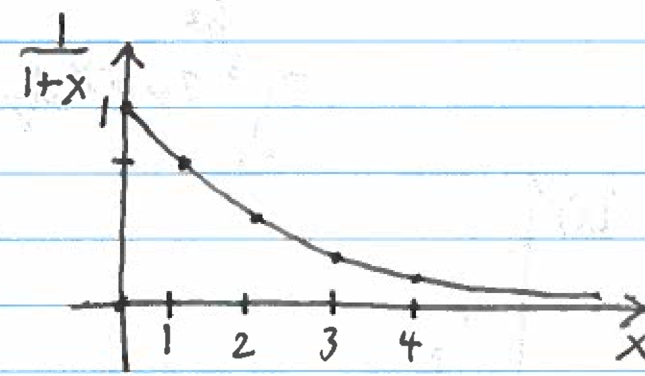
Example:

If X is the number of heads obtained in four tosses of a balanced coin, find the probability distribution of

1. $Y = \dfrac{1}{1+X}$,
2. $Z = (X-2)^2$

$$-\; p(x) = \binom{4}{x}(1/2)^x(1-1/2)^{4-x}$$
$$= \binom{4}{x}\left(\tfrac{1}{2}\right)^4.$$

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



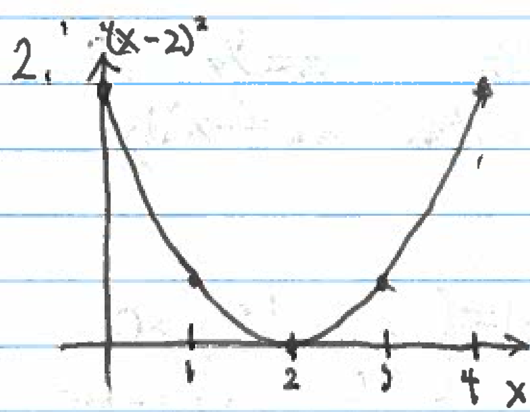
1.

y	1	$1/2$	$1/3$	$1/4$	$1/5$
$q(y)$	$\frac{1}{16}$	$4/16$	$6/16$	$4/16$	$1/16$

$$Y = \frac{1}{1+X} \Rightarrow X = \frac{1}{Y} - 1$$

$$\Rightarrow q(y) = p(x(y)) \qquad (\text{for } y = 1, 1/2, 1/3, 1/4, 1/5)$$
$$= \binom{4}{\frac{1}{y}-1}\left(\tfrac{1}{2}\right)^4$$

2.



z	0	1	4
$r(z)$	$6/16$	$8/16$	$2/16$

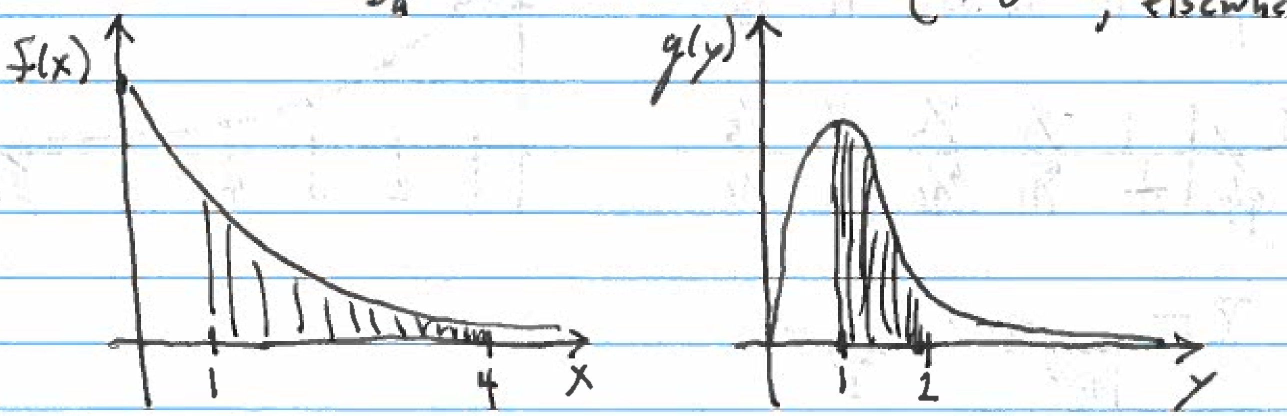
Example:

If X has the exponential distribution with density

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

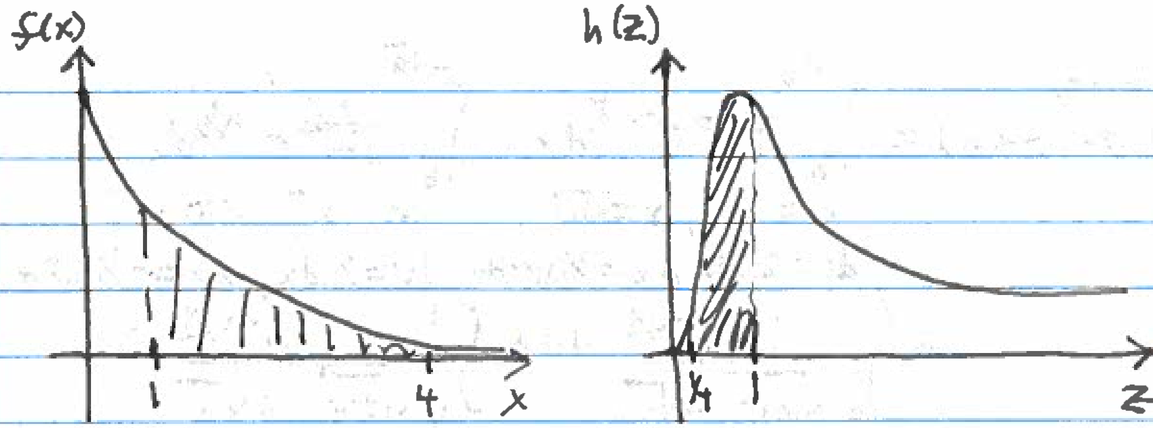
find the probability density of $Y = \sqrt{X}$, $Z = 1/X$.

$$P(a < Y < b) = P(a < \sqrt{X} < b)$$
$$= P(a^2 < x < b^2)$$
$$= \int_{a^2}^{b^2} e^{-x}\,dx \qquad y = \sqrt{x}, \quad x = y^2$$
$$dy = \tfrac{1}{2} x^{-1/2} dx \quad dx = 2x^{1/2} dy$$
$$= \int_a^b 2y e^{-y^2} dy \Rightarrow g(y) = \begin{cases} 2y e^{-y^2}, & 0 < y \\ 0, & \text{elsewhere} \end{cases}$$



(Areas are squished by the transformation)

$$P(a < Z < b) = P(a < 1/X < b)$$
$$= P(1/b < X < 1/a)$$
$$= \int_{1/b}^{1/a} e^{-x}\,dx \qquad y = 1/x, \quad x = 1/y$$
$$dy = -1/x^2 dx \quad dx = -x^2 dy$$
$$= \int_b^a -x^2 e^{-1/y} dy$$
$$= \int_a^b \frac{1}{y^2} e^{-1/y} dy \Rightarrow h(z) = \begin{cases} z^{-2} e^{-1/z}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$



(Transformed areas).

Example:

If X has the standard normal distribution, find the density of $Z = X^2$.

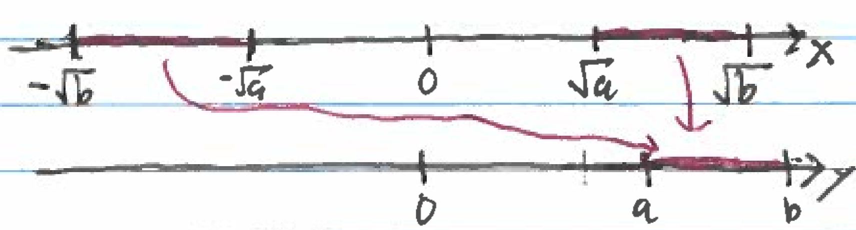
$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$P(a < Z < b) = \begin{cases} 0, & b < 0 \\ P(0 < Z < b), & a < 0 \\ P(a < Z < b), & a > 0 \end{cases}$$

Now, $X = \pm\sqrt{Z}$ since the functional relationship is not invertible

$\Rightarrow$ If $a > 0$ then

$$P(a < Z < b) = P(a < X^2 < b)$$
$$= P(\sqrt{a} < X < \sqrt{b}) + P(-\sqrt{b} < X < -\sqrt{a})$$



Therefore,

$$P(a<Z<b) = \int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_{-\sqrt{b}}^{-\sqrt{a}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$u = x^2$, $x = \sqrt{u}$ $\qquad$ $v = x^2$, $x = -\sqrt{v}$

$du = 2x\,dx$ $dx = \frac{1}{2\sqrt{u}} du$ $\qquad$ $dv = 2x\,dx$, $dx = -\frac{1}{2\sqrt{v}} dv$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} \frac{e^{-u/2}}{2\sqrt{u}} du + \int_b^a \frac{-1}{\sqrt{2\pi}} \frac{e^{-v/2}}{2\sqrt{v}} dv$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} \frac{e^{-z/2}}{\sqrt{z}} dz$$

$$\Rightarrow f(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-z/2}, & z > 0 \\ 0, & \text{elsewhere.} \end{cases}$$