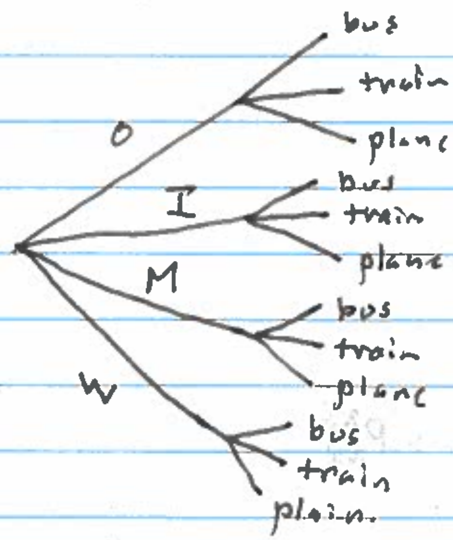
Lecture 3: Counting and Probability

- S has N elements: $\{x_1, \ldots, x_N\}$ and $P(x_i) = P(x_j)$ for all i, j. Then

$$P(A) = \frac{\#\text{ of elements in } A}{N}.$$

Example:

Suppose that someone wants to go by bus, train, or plane to Ohio, Indiana, Michigan, or Wisconsin. How many ways can this be done.



$\Rightarrow$ 12 different ways.

Example:

How many ways can one answer all the questions of a true-false test containing 20 questions

$$2 \cdot 2 \cdot \ldots \cdot 2 = 2^{20} = 1{,}048{,}576$$

Example:

In how many ways can the five starting players of a basketball team be introduced to the public?

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120 \text{ ways}$$

Example:

How many ways can 4 people playing hearts be arranged

$3 \cdot 2 \cdot 1 = 3! = 6.$

Example:

Using the English alphabet, how many 3 letter words are possible that

(a) are allowed to repeat letters

(b) are not allowed to repeat letters

(c) can repeat one letter

(a): $26 \cdot 26 \cdot 26 = 26^3$

(b): $26 \cdot 25 \cdot 24 = \frac{26!}{23!} = P^{26}_{3}$

$* \; P^n_r = \frac{n!}{(n-r)!}$

(c): $\frac{26!}{23!} + 26 \cdot 25 = P^{26}_{24} + P^{26}_{25}$

no repeats one repeat

(d) What is the probability a random 3 letter word does not repeat a letter.

$$\frac{\frac{26!}{23!}}{26^3} = \frac{25 \cdot 24}{26^2} = .89.$$

Example:

How many different permutations of the word book are there?

Label as $b o_1 o_2 k$ and then we have $4!$ permutations however since o_1 and o_2 can be switched we have

$$\# \text{of arrangements} = \frac{4!}{2} = 12.$$

Example:

How many different ways can three copies of one novel and one copy each of four other novels be arranged on a shelf?

Label first novel as a_1, a_2, a_3 other novels as b, c, d, e. Then we can arrange these in $7!$ ways but a_1, a_2, a_3 can be rearranged in $3!$ ways so

$$\# \text{of arrangements} = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

Example:

How many ways can two paintings by Monet, three paintings by Renoir, and two by Degas be hung side by side in a museum if we not distinguish by artist?

Label paintings by $m_1, m_2, m_3, d_1, d_2, r_1, r_2, r_3$

$$\# \text{of arrangements} = \frac{7!}{3!\,2!\,3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{3! \cdot 2!} = 7 \cdot 5 \cdot 2 = 70.$$

Example:

Four names are drawn from 24 members of a club for the offices of president, vice president, treasurer, and secretary. In how many different ways can this be done?

$$P^{24}_{4} = \frac{24!}{20!} = 24 \cdot 23 \cdot 22 \cdot 21 = 255{,}024.$$

Example:

In how many ways can a person gathering data for a market research organization select three of twenty households living in a certain apartment complex?

$$P^{20}_{3} = 20 \cdot 19 \cdot 18 = 6{,}840 = \text{\# of ways if order matters.}$$

but if order does not matter we have counted an additional $3! = 6$ for each combination of houses

$$\Rightarrow \text{\# of ways} = \frac{6840}{6} = 1{,}140.$$

Theorem - The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{P^{n}_{r}}{r!} = \frac{n!}{r!(n-r)!}$$

Example:

How many different ways can six coin tosses of a coin yield two heads and four tails?

This is the same as choosing two slots at which heads occurs

$$\Rightarrow \binom{6}{2} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5}{2} = 15. \text{ ways.}$$

Example:

How many different committees of two chemists and one physicist can be formed from the four chemists and three physicists on the faculty of a small college?

$$\binom{4}{2}\binom{3}{1} = \frac{4!}{2!2!} \cdot \frac{3!}{1! \cdot 2!} = 6 \cdot 3 = 18$$

arrangement of chemists

arrangement of physicists

Example:

In how many ways can seven students be assigned to one triple and two double hotel rooms?

Label students as $s_1 s_2 s_3 s_4 s_5 s_6 s_7$.

$\rightarrow$ $s_1 s_2 s_3 | s_4 s_5 | s_6 s_7$ one possible arrangement.

all arrangements in order matters

$$\frac{7!}{3!2!2!} = \binom{7}{3,2,2} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4} = 42 \cdot 5 = 210.$$

# of ways to arrange room 1

# of ways to arrange room 2

# of ways to arrange room 3