Lecture 9: Expected Value and Variance

Definition - Let X be a discrete random variable with probability distribution $p(x)$. The expected value of X, $\mathbb{E}[X]$, is defined by

$$\mathbb{E}[X] = \sum_x x p(x).$$ ~ weighted average.

Example:

Return to the sock example
- 5 brown socks
- 3 green socks
- two socks selected at random

$X = \#$ of brown socks

$p(0) = P(X=0) = 6/56$
$p(1) = P(X=1) = 30/56$
$p(2) = P(X=2) = 20/56$

$$\Rightarrow \mathbb{E}[X] = 0 \cdot p(0) + 1\,p(1) + 2p(2) = 30/56 + 40/56 = 70/56 = 5/4 = 1.25$$

Definition - If $\mathbb{E}[X] = \mu$, the variance of a random variable X is given by

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x).$$

σ = standard deviation

Example:

Returning to the socks example:

$$\mathbb{V}[X] = (-1.25)^2\, 6/56 + (1-1.25)^2\, 30/56 + (2-1.25)^2\, 20/56$$
$$= .4018$$
$$\Rightarrow \sigma = \sqrt{\mathbb{V}[X]} = .63$$

Theorem - $\mathbb{E}[ag(X)+bh(X)+c] = a\mathbb{E}[g(X)]+b\mathbb{E}[h(X)]+c$

proof:

$$\mathbb{E}[ag(X)+bh(X)+c] = \sum_x (ag(X)+bh(X)+c)p(x)$$

$$= a\sum_x g(X)p(x) + b\sum_x h(X)p(x) + c\sum_x p(x)$$

$$= a\mathbb{E}[g(X)]+b\mathbb{E}[h(X)]+c.$$

Example:

X = roll of six side dic.

$$\mathbb{E}[2X^2+1] = 2\mathbb{E}[X^2]+1 = 2(1+4+9+16+25+36)\tfrac{1}{6}+1 = \tfrac{94}{3} = 31.33.$$

Theorem - $\mathbb{V}[X] = \mathbb{E}[X^2]-\mathbb{E}[X]^2 = \mathbb{E}[X^2]-\mu^2.$

proof:

$$\mathbb{V}[X] = \mathbb{E}[(X-\mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2-2X\mathbb{E}[X]+\mathbb{E}[X]^2]$$
$$= \mathbb{E}[X^2]-2\mathbb{E}[X\mathbb{E}[X]]+\mathbb{E}[X]^2$$
$$= \mathbb{E}[X^2]-2\mathbb{E}[X]\mathbb{E}[X]+\mathbb{E}[X]^2$$
$$= \mathbb{E}[X^2]-\mathbb{E}[X]^2.$$