Linear Algebra II
Spring 2024
Exam 1
02/16/24

Name (Print): Key

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	10	
7	15	
Total:	100	

Do not write in the table to the right.

1. (15 points) For what values of $a \in \mathbb{R}$ does the following system of equations have zero, one, or infinitely many solutions?

$$\begin{aligned} x + y + z &= 4, \\ y + z &= 2, \\ (a^2 - 4)z &= a - 2. \end{aligned}$$

- If $a=2$ there are infinitely many solutions
- If $a=-2$ there are no solutions
- If $a \neq 2$ or $a \neq -2$ there is one solution.

2. (15 points)

(a) (5 points) **Short Answer:** Write down the two properties a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ must satisfy in order for them to form a basis for a vector space V.

1. $\vec{v}_1, \ldots, \vec{v}_n$ are linearly ind.
2. $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V$

(b) (10 points) Determine if the following set of polynomials forms a basis for P_2:

$$\{1 + x + x^2, -1 + x + x^2, 1 + x - x^2\}.$$

$$c_1(1+x+x^2) + c_2(-1+x+x^2) + c_3(1+x-x^2) = 0$$

$$\Rightarrow \begin{matrix} c_1 - c_2 + c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 - c_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \begin{matrix} \\ -R1 \\ -R1 \end{matrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \begin{matrix} \\ \\ -R2 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$\Rightarrow$ $1+x+x^2, -1+x+x^2, 1+x-x^2$ are linearly ind.
Since $\dim(\text{span}\{1+x+x^2, -1+x+x^2, 1+x-x^2\}) = 3$ it follows that $\{1+x+x^2, -1+x+x^2, 1+x-x^2\}$ is a basis for $P_2(\mathbb{R})$.

3. (15 points)

(a) (5 points) **Short Answer:** Let V be a vector space and $U \subset V$. Write down the properties U must satisfy in order for U to be a subspace of V.

1. $0 \in U$
2. For all $\vec{u}, \vec{v} \in V$, $\vec{u} + \vec{v} \in V$
3. For all $k \in F$ and $\vec{u} \in V$, $k\vec{u} \in V$.

(b) (10 points) Let W be the set of 2×2 matrices of the form

$$W = \begin{bmatrix} a & a+c \\ b-c & b \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$. Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$.

1. If $a = b = c = 0$, then

$$\begin{bmatrix} a & a+c \\ b-c & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} a_1 & b_1 + c_1 \\ b_1 - c_1 & b_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 + c_2 \\ b_2 - c_2 & b_2 \end{bmatrix}$.

$$\Rightarrow A + B = \begin{bmatrix} a_1 + a_2 & (b_1+b_2) + (c_1+c_2) \\ (b_1+b_2) - (c_1+c_2) & b_1 + b_2 \end{bmatrix} \in W$$

3. Let $k \in F$

$$\Rightarrow k\begin{bmatrix} a & a+c \\ b-c & b \end{bmatrix} = \begin{bmatrix} ka & k(a+c) \\ k(b-c) & kb \end{bmatrix} = \begin{bmatrix} ka & ka+kc \\ kb-kc & kb \end{bmatrix} \in W$$

4. (15 points)

(a) (5 points) **Short Answer:** Let V and W be vector spaces. Write down the two properties a function $T : V \mapsto W$ must satisfy in order for it to be a linear transformation.

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(k\vec{u}) = kT(\vec{u})$.

(b) (10 points) Let $T : \mathbb{R}^2 \mapsto P_2(\mathbb{R})$ be a linear transformation satisfying

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = x^2 + x \text{ and } T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = x^2 - x + 1.$$

Find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$.

$$\begin{bmatrix}1\\0\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \tfrac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{1}{2}(x^2+x) + \frac{1}{2}(x^2-x+1)$$

$$= x^2 + \tfrac{1}{2}.$$

5. (15 points) Let $T : \mathbb{R}^5 \mapsto \mathbb{R}^2$ be a linear transformation

   (a) (3 points) **Short Answer:** What is the largest possible value of $\dim(\ker(T)) + \dim(\mathrm{im}(T))$?

   5

   (b) (3 points) **Short Answer:** What is the largest possible value of $\dim(\ker(T))$?

   5

   (c) (3 points) **Short Answer:** What is the smallest possible value of $\dim(\ker(T))$?

   3

   (d) (3 points) **Short Answer:** What is the largest possible value of $\dim(\mathrm{im}(T))$?

   2

   (e) (3 points) **Short Answer:** What is the smallest possible value of $\dim(\mathrm{im}(T))$?

   0

6. (10 points) Suppose that $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ forms a basis for a vector space V and $\beta = \{\vec{w}_1, \vec{w}_2\}$ forms a basis for a vector space W. Suppose $T : V \mapsto W$ is a linear transformation satisfying

$$T(\vec{v}_1) = 2\vec{w}_1 - 3\vec{w}_2$$
$$T(\vec{v}_2) = -\vec{w}_1 + 3\vec{w}_2$$
$$T(\vec{v}_3) = \vec{w}_1 + 2\vec{w}_2$$
$$T(\vec{v}_4) = 3\vec{w}_2$$

Short Answer: Find $T[\alpha, \beta]$.

$$[T(\alpha, \beta)] = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -3 & 3 & 2 & 3 \end{bmatrix}.$$

7. (15 points)

(a) (5 points) **Short Answer:** Let V be a vector space and $T : V \mapsto V$ a linear transformation. Write down what it means for $\vec{v} \in V$ to be an eigenvector of T with eigenvalue λ.

$T(\vec{v}) = \lambda \vec{v}.$

(b) (10 points) Suppose $T : V \mapsto V$ is a linear transformation and $\vec{u}, \vec{v}$ and $\vec{u} + \vec{v}$ are eigenvectors of T with corresponding eigenvalues λ_1, λ_2, λ_3. If $\vec{u}$ and $\vec{v}$ are linearly independent, show that $\lambda_1 = \lambda_2 = \lambda_3$.

$T(\vec{u}+\vec{v}) = \lambda_3(\vec{u}+\vec{v})$

$\Rightarrow T(\vec{u}) + T(\vec{v}) = \lambda_3 \vec{u} + \lambda_3 \vec{v}$

$\Rightarrow \lambda_1 \vec{u} + \lambda_2 \vec{v} = \lambda_3 \vec{u} + \lambda_3 \vec{v}$

$\Rightarrow (\lambda_1 - \lambda_3)\vec{u} + (\lambda_2 - \lambda_3)\vec{v} = 0$

$\Rightarrow \lambda_1 = \lambda_3$

$\lambda_2 = \lambda_3$