MTH 225: Homework #7

Due Date: March 22, 2024

1. Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ are diagonalizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

2. Compute a unitary diagonalization of each of the following Hermitian matrices (give the diagonal matrix and the unitary matrix) and give the spectral decomposition:

$$A = \begin{bmatrix} 7 & i \\ -i & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1-i \\ 1+i & 0 \end{bmatrix}.$$

3. Consider the following vectors in $\mathbb{C}^4$:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix}, \qquad \vec{w}_2 = \begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix}, \qquad \vec{w}_3 = \begin{bmatrix} -1 \\ i \\ -i \\ 1 \end{bmatrix}.$$

   (a) Show that $\langle \vec{w}_1, \vec{w}_2 \rangle = 0$.

   (b) Find an orthonormal basis of the subspace $W = \operatorname{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ of $\mathbb{C}^4$.

4. Prove the converse of the Spectral Theorem: If $A = UDU^*$ for a unitary matrix U and a diagonal matrix D, whose entries are all real numbers, then A must be a Hermitian matrix.

5. Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{C}^n$.

   (a) Find a nonzero eigenvalue of $\vec{u}\vec{v}^*$, and determine its corresponding eigenvector.

   (b) Determine the unique nonzero singular value σ of $\vec{u}\vec{v}^*$, as well as the corresponding singular vectors $\vec{u}_1$ and $\vec{v}_1$ corresponding to σ.

6. Let $\{\vec{u}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Prove that for any $\vec{v} \in \mathbb{C}^n$, one has the equality

$$\|\vec{v}\|^2 = \sum_{j=1}^{n} |\langle \vec{u}_j, \vec{v} \rangle|^2.$$

   Hint: Use the projection formula to express $\vec{v}$ as a linear combination of the given basis.

7. A matrix $P \in M_{n\times n}(\mathbb{C})$ is called a projection matrix if $P^2 = P$.

   (a) Show that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix and $P\vec{v} \neq \vec{v}$ then $P\vec{v} - \vec{v} \in \ker(A)$.

   (b) Prove that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix then $I - P$ is also a projection matrix.

   (c) If $\vec{q} \in \mathbb{C}^n$ satisfies $\|\vec{q}\| = 1$, prove that $\vec{q}\vec{q}^*$ is a projection matrix.

   (d) If P is projection matrix, prove the following three statements

$$\operatorname{im}(I - P) = \ker(P),$$
$$\operatorname{im}(P) = \ker(I - P),$$
$$\operatorname{im}(P) \cap \ker(P) = \{0\}.$$

8. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Find the singular value decomposition of A, B, C. You will probably have to use the Gram Matrix to compute these decompositions.

(b) Compute the closest rank 1 matrices to A, B, and C.

Homework #7

#2.

Compute a unitary diagonalization of the following Hermitian matrices and give the spectral decomposition.

$$A=\begin{bmatrix} 7 & i \\ -i & 7 \end{bmatrix},\quad B=\begin{bmatrix} 1 & 1-i \\ 1+i & 0 \end{bmatrix}.$$

Solution:

(a) $\det(\lambda I-A)=\det\left(\begin{bmatrix} \lambda-7 & -i \\ i & \lambda-7 \end{bmatrix}\right)=(\lambda-7)^2-1$

$\Rightarrow \lambda=8,6$

$\lambda_1=8$:

$$\lambda_1 I-A=\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \Rightarrow \vec{v}_1=\begin{bmatrix} i \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1=\begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$\lambda_2=6$:

$$\lambda_2 I-A=\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \Rightarrow \vec{v}_2=\begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow \vec{v}_2=\begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$=8\begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}+6\begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(b) $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & -1+i \\ -1-i & \lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

$\Rightarrow \lambda = 2, -1$

$\underline{\lambda_1 = 2:}$

$\lambda_1 I - A = \begin{bmatrix} 1 & -1+i \\ -1-i & 2 \end{bmatrix} \Rightarrow \tilde{v}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} (1-i)/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\underline{\lambda_2 = -1:}$

$\lambda_2 I - A = \begin{bmatrix} -2 & -1+i \\ -1-i & -1 \end{bmatrix} \Rightarrow \tilde{v}_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1/\sqrt{3} \\ (-1-i)/\sqrt{3} \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} (1-i)/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & (-1-i)/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} (1+i)/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & (-1+i)/\sqrt{3} \end{bmatrix}$

$\Rightarrow A = 2 \begin{bmatrix} (1-i)/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} (1+i)/\sqrt{3} & \sqrt{3} \end{bmatrix} - 1 \begin{bmatrix} 1/\sqrt{3} \\ (-1-i)/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & (-1+i)/\sqrt{3} \end{bmatrix}$

$\blacksquare$

#3.

Consider the following vectors in $\mathbb{C}^4$:

$\vec{w}_1 = \begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} -1 \\ i \\ -i \\ 1 \end{bmatrix}.$

(a) Show that $\langle \vec{w}_1, \vec{w}_2 \rangle = 0$

(b). Find an orthonormal basis of the subspace $W = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

<u>Solution:</u>

(a) $\langle \vec{w}_1, \vec{w}_2 \rangle = \vec{w}_1^* \vec{w}_2 = \begin{bmatrix} 1 & -i & i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix} = i - i + i - i = 0.$

(b). $\|\vec{w}_1\| = 2$, $\|\vec{w}_2\| = 2$.

Therefore,

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ i/2 \\ -i/2 \\ -1/2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} i/2 \\ 1/2 \\ 1/2 \\ -i/2 \end{bmatrix}$$

Computing we have that

$$\tilde{v}_3 = \vec{w}_3 - \langle \vec{w}_3, \vec{v}_1 \rangle \vec{v}_1 - \langle \vec{w}_3, \vec{v}_2 \rangle \vec{v}_2$$

Now,

$$\langle \vec{w}_3, \vec{v}_1 \rangle = \begin{bmatrix} -1 & -i & i & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ i/2 \\ -i/2 \\ -1/2 \end{bmatrix} = -1/2 + 1/2 + 1/2 + 1/2 = 1$$

$$\langle \vec{w}_3, \vec{v}_2 \rangle = \begin{bmatrix} -1 & -i & i & 1 \end{bmatrix} \begin{bmatrix} i/2 \\ 1/2 \\ 1/2 \\ -i/2 \end{bmatrix} = -i/2 - i/2 + i/2 - i/2 = -i.$$

Consequently,

$$\hat{v}_3 = \begin{bmatrix} -1 \\ i \\ -i \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1/2 \\ i/2 \\ -i/2 \\ -1/2 \end{bmatrix} + i \begin{bmatrix} i/2 \\ i/2 \\ 1/2 \\ -i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$\blacksquare$

#4.

Prove the converse of the spectral theorem.

Solution:

Suppose $A = UDU^*$ for a unitary matrix U and a diagonal matrix D, whose entries are all real numbers. Therefore,

$$\begin{aligned} A^* &= (UDU^*)^* \\ &= (U^*)^* D^* U^* \\ &= UDU^*. \end{aligned}$$

Therefore, A is unitary. $\blacksquare$

#5

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{C}^n$.

(a) Find a nonzero eigenvalue of $\vec{u}\vec{v}^*$, and determine its corresponding eigenvector.

(b) Determine the unique nonzero singular value σ of $\vec{u}\vec{v}^*$, as well as the corresponding singular vectors $\vec{u}_1$ and $\vec{v}_1$.

Solution:

(a) Since $\operatorname{im}(\vec{u}\vec{v}^*) = \operatorname{span}(\{\vec{u}\})$ it follows that $\vec{u}$ is an eigenvector. Moreover, $\vec{u}\vec{v}^*\vec{u} = (\vec{v}^*\vec{u})\vec{u}$ and thus $\lambda = \vec{v}^*\vec{u}$.

(b) Since $\operatorname{im}(\vec{u}\vec{v}^*) = \operatorname{span}(\{\vec{u}\})$ it follows that $\vec{u}_1 = \vec{u}/\|\vec{u}\|$. Moreover, since for all $\vec{w}$, $\vec{u}\vec{v}^*\vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)\vec{u}$, where θ is the angle between $\vec{v}$ and $\vec{w}$, it follows that the greatest stretch occurs when $\vec{v}$ and $\vec{w}$ are parallel. Consequently, $\vec{v}_1 = \vec{v}/\|\vec{v}\|$. Now,

$$\sigma_1 \vec{u}_1 = \vec{u}\vec{v}^* \frac{\vec{v}}{\|\vec{v}\|} = \vec{u}\|\vec{v}\| = \frac{\vec{u}\|\vec{u}\| \cdot \|\vec{v}\|}{\|\vec{u}\|}$$

and therefore $\sigma_1 = \|\vec{u}\| \cdot \|\vec{v}\|$.

$\blacksquare$

#6.

Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Prove that for any $\vec{v} \in \mathbb{C}^n$, one has the equality

$$\|\vec{v}\|^2 = \sum_{j=1}^{n} |\langle \vec{u}_j, \vec{v} \rangle|^2.$$

~~proof:~~

$$\vec{v} = \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 + \langle \vec{u}_2, \vec{v} \rangle \vec{u}_2 + \dots + \langle \vec{u}_{n-1}, \vec{v} \rangle \vec{u}_{n-1} + \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n$$

$$\Rightarrow \|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$$

$$= \langle \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 + \dots + \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n, \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 + \dots + \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n \rangle$$

$$= \langle \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1, \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 \rangle + \langle \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1, \langle \vec{u}_2, \vec{v} \rangle \vec{u}_2 \rangle + \dots + \langle \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1, \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n \rangle$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$+ \langle \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n, \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 \rangle + \langle \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n, \langle \vec{u}_2, \vec{v} \rangle \vec{u}_2 \rangle + \dots + \langle \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n, \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n \rangle$$

$$\Rightarrow \|\vec{v}\|^2 = \langle \vec{v}_1, \vec{v}\rangle \overline{\langle \vec{v}_1, \vec{v}\rangle} \cancelto{1}{\langle \vec{v}_1, \vec{v}_1\rangle} + \langle \vec{v}_1, \vec{v}\rangle \overline{\langle \vec{v}_2, \vec{v}\rangle} \cancelto{0}{\langle \vec{v}_1, \vec{v}_2\rangle} + \dots + \langle \vec{v}_1, \vec{v}\rangle \overline{\langle \vec{v}_n, \vec{v}\rangle} \cancelto{0}{\langle \vec{v}_1, \vec{v}_n\rangle}$$

$$\vdots$$

$$+ \langle \vec{v}_n, \vec{v}\rangle \overline{\langle \vec{v}_1, \vec{v}\rangle} \cancelto{0}{\langle \vec{v}_n, \vec{v}_1\rangle} + \langle \vec{v}_n, \vec{v}\rangle \overline{\langle \vec{v}_2, \vec{v}\rangle} \cancelto{0}{\langle \vec{v}_n, \vec{v}_2\rangle} + \dots + \langle \vec{v}_n, \vec{v}\rangle \overline{\langle \vec{v}_n, \vec{v}\rangle} \cancel{\langle \vec{v}_n, \vec{v}_n\rangle}$$

$$= |\langle \vec{v}_1, \vec{v}\rangle|^2 + |\langle \vec{v}_2, \vec{v}\rangle|^2 + \dots + |\langle \vec{v}_n, \vec{v}\rangle|^2$$

∎

#7

A matrix $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix if $P^2 = P$.

(a) Show that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix and $P\vec{v} \neq \vec{v}$ then $P\vec{v} - \vec{v} \in \ker(P)$.

(b) Prove that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix then $I - P$ is also a projection matrix.

(c) If $\vec{q} \in \mathbb{C}^n$ satisfies $\|\vec{q}\| = 1$, prove that $\vec{q}\vec{q}^*$ is a projection matrix.

(d) If P is a projection matrix, prove the following three statements

$$\operatorname{im}(I - P) = \ker(P)$$
$$\operatorname{im}(P) = \ker(I - P)$$
$$\operatorname{im}(P) \cap \ker(P) = \{0\}.$$

Solution:

(a) $P(P\vec{v} - \vec{v}) = P^2\vec{v} - P\vec{v} = P\vec{v} - P\vec{v} = 0.$

(b) $(I - P)^2 = I^2 - 2IP + P^2 = I - 2P + P = I - P$

(c) $(\vec{q}\vec{q}^*)(q q^*) = \vec{q}\,\vec{q}^* q\, q^* = q^* q \cdot q q^* = q q^*.$

(d) - $\vec{v} \in \ker(P) \Leftrightarrow P\vec{v} = 0 \Leftrightarrow P\vec{v} = \vec{v} - \vec{v} \Leftrightarrow \vec{v} = \vec{v} - P\vec{v} \Leftrightarrow \vec{v} = (I - P)\vec{v}.$

- Let $\tilde{P} = I - P$ and thus $I - \tilde{P} = P$. Therefore, since

$$\ker(\tilde{P}) = \operatorname{im}(I - \tilde{P})$$

we have that

$$\ker(I - P) = \operatorname{im}(P).$$

- If we let $\vec{v} \in im(P)$ and $\vec{v} \in ker(P)$ then there exists $\vec{u}$ such that $P\vec{u} = \vec{v}$ and $P\vec{v} = 0$. Consequently,

$$P^2\vec{u} = P\vec{v} = 0$$

$$\Rightarrow P\vec{u} = 0$$

$$\Rightarrow \vec{v} = 0$$

$\blacksquare$